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FIXED POINT ALGEBRAS OF UHF-ALGEBRAS III

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1. Introduction

In this note we give some concrete examples of C^*-dynamical systems ($\mathfrak{A}$, G, α) where $\mathfrak{A}$ is a uniformly hyperfinite (UHF) algebra and G is a finite group and α is the *-automorphic action of G of product type. We consider the fixed point algebra

$$\mathfrak{A}^\alpha = \{x \in G : \alpha_g(x) = x \text{ for all } g \in G\}.$$

In our situation, $\mathfrak{A}^\alpha$ is always approximately finite dimensional (AF) and simple (for the detail, see [1, 2, 3]).

UHF-algebras are simple and of the unique tracial state. On the contrary, AF-algebra is not so in general. Here we construct a example such that $\mathfrak{A}^\alpha$ is simple and it has two extremal tracial states. Another example $\mathfrak{A}^\alpha$ is a *non* UHF-algebra with a unique tracial state. The last example is that $\mathfrak{A}^\alpha$ is UHF.

2. Preliminaries and notations

Let K_i, $i \in \mathbb{N}$ be matrix algebras of rank $|K_i|$, that is, $|M_n(\mathbb{C})| = n$. Here $M_n(\mathbb{C})$ is the all $n \times n$ complex matrices. The UHF-algebra $\mathfrak{A}$ is defined by the infinite tensor product of K_i's

$$\mathfrak{A} = \bigotimes_{i=1}^{\infty} K_i.$$

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Let G be a finite group and consider unitary representations $\pi_i : G \to K_i$ for each $i \in \mathbb{N}$. Then we can define the homomorphism α from G into all *-automorphisms of $\mathfrak{A}$ such that

$$\alpha_g = \bigotimes_{i=1}^{\infty} Ad\pi_i(g) \quad \text{where} \quad Ad\pi_i(g) = \pi_i(g) \cdot \pi_i(g)^*.$$

We obtain a C^*-dynamical system $(\mathfrak{A}, G, \alpha)$. We assume throughout that α_g are not inner in $\mathfrak{A}$ when $g \neq e$ where e is the unit of G. Let τ be the unique tracial state on $\mathfrak{A}$ defined by

$$\tau = \bigotimes_{i=1}^{\infty} |K_i|^{-1} Tr$$

where Tr is the usual trace on the matrix algebra K_i (here we did not write as Tr_i in order to simplify notations without confusion.)

As same as in [3], we can extend the C^*-dynamical system $(\mathfrak{A}, G, \alpha)$ to the W^*-dynamical system $(\pi_\tau(\mathfrak{A})'', G, \widetilde{\alpha})$ since the trace τ is α-invariant. Here π_τ is the G.N.S.-representation of $\mathfrak{A}$ constructed by τ and $\pi_\tau(\mathfrak{A})''$ is the hyperfinite factor of type II_1. We set

$$K = \{g \in G : \widetilde{\alpha}_g \text{ is an inner automorphism of } \pi_\tau(\mathfrak{A})''\}.$$

Let $\widehat{K}$ be the dual object of K. Since K is a normal subgroup of G, we get a G-space $(G, \widehat{K})$ with the action $(g \cdot \pi)(k) = \pi(gkg^{-1})$ for all $g \in G$, $k \in K$ and $\pi \in \widehat{K}$. We define a G-space $\widehat{K}/G$, i.e., all G-orbits in $\widehat{K}$. In our situation, the fixed point algebra $\mathfrak{A}^\alpha$ is simple, i.e., has no proper closed ideal. For the detail, see [3].

Put $W_g^{n,m} = \bigotimes_{i=n+1}^{m} \pi(g)$, $n < m$. Since $W^{n,m}$ is a unitary representation of G into $\bigotimes_{i=n+1}^{m} K_i$, we get an irreducible decomposition $W^{n,m} = \sum_{\pi \in \widehat{G}} \lambda_\pi^{n.m} \pi$ where $\lambda_\pi^{n,m}$ is the multiplicity of π in $W^{n,m}$. We define a positive element in $\bigotimes_{i=n+1}^{m} K_i$

$$E_{\rho,\overline{\pi}}^{n,m} = \int_G \overline{\chi_\rho(g)} \chi_\pi(g) W_g^{n,m}\, dg \quad \text{for} \quad \rho, \pi \in \widehat{G}$$

where χ_π is the character with respect to π and dg is a normalized Haar measure on G.

As [1, 2, 3, 4],

$$\mathfrak{A}^\alpha = \overline{\bigcup_{n=1}^{\infty} \mathfrak{A}_n^\alpha} \quad \text{where} \quad \mathfrak{A}_n = \bigotimes_{i=1}^{n} K_i.$$

Here $\overline{\{\ \ \}}$ denotes the norm closure. Then the finite dimensional algebra $\mathfrak{A}_n^\alpha = \mathfrak{A}_n \cap \{W_g^{0,n} : g \in G\}'$ is isomorphic to $\bigoplus_{\pi \in \widehat{G}} \mathfrak{A}_\pi^n$ where $\mathfrak{A}_\pi^n$ is a factor of type $I_{\lambda_\pi^{0,n}}$. Hence the AF-algebra $\mathfrak{A}^\alpha$ is completely determined by the partial embedding $\mathfrak{A}_\pi^n \to \mathfrak{A}_\rho^{n+1}$ with the multiplicity $|K_{n+1}|\tau(E_{\rho,\overline{\pi}}^{n,n+1})$ [3. Lemma 2.1]. Then the followings were obtained in [3].

PROPOSITION 1. [3, Theorem 3.1] *Let* ($\mathfrak{A}$, G, α) *and* K *be as above. Then the number of all extremal tracial states on* $\mathfrak{A}^\alpha$ *equals the cardinality of the orbit space* $\widehat{K}/G$.

PROPOSITION 2. [3, Theorem 3.6] *Let* ($\mathfrak{A}$, G, α) *be as above. Then* $\mathfrak{A}^\alpha$ *is UHF if and only if there exists an increasing sequence* $\{n_k : k \in \mathbb{N}\}$ *such that* $n_1 = 0$ *and*

$$\tau(E_{\rho,\overline{\pi}}^{n_k,n_{k+1}}) = |G|^{-1} \dim\rho \dim\pi \text{ for all } \rho, \pi \in \widehat{G} \text{ and all } k \in \mathbb{N}.$$

Here $|G|$ *is the cardinality of* G.

3. Main results and examples

Here we investigate that there exist a simple AF-algebra with two tracial states (Example 1), a simple AF-algebra (but *non* UHF) with a unique tracial state (Example 2) and a UHF fixed point algebra (Example 3) using Proposition 1 and 2.

EXAMPLE 1. et $G = \mathcal{S}(3)$ be the symmetric group of three elements. It is well known that $\mathcal{S}(3)$ has two one-dimensional irreducible representations ι and sgn, and one two-dimensional irreducible representation π. Let K_n be the algebra of all $(n^2 + n^2 + 2) \times (n^2 + n^2 + 2)$ complex matrices and π_n be the representation of $G = \mathcal{S}(3)$ into K_n such that

$$\pi_n = n^2\iota \oplus n^2 sgn \oplus \pi.$$

Then we have

$$\frac{1}{2n^2+2}Tr(\pi_n(g)) = \begin{cases} 1 & , \text{ if } g=e \\ \dfrac{n^2-n^2+1\cdot 0}{2n^2+2} = 0, & \text{if } g=(1,2),(1,3) \text{ or } (2,3) \\ \dfrac{2n^2-1}{2n^2+2} & , \text{ if } g=(1,2,3) \text{ or } (1,3,2). \end{cases}$$

Since the normal subgroup K of $G = \mathcal{S}(3)$ is

$$\left\{g \in \mathcal{S}(3) : \sum_{n=1}^{\infty}\{1-\tau(\pi_n(g))\} < \infty\right\},$$

K is the alternating subgroup $\mathcal{A}(3)$ of $\mathcal{S}(3)$. By an easy computation, the dual object $\widehat{\mathcal{A}(3)}$ of $\mathcal{A}(3)$ consists of three points and the orbit space $\widehat{\mathcal{A}(3)}/\mathcal{S}(3)$ consists of two orbits. Therefore this fixed point algebra is simple but has two extremal tracial states by Proposition 1.

PROPOSITION 3. *There exists a simple AF-algebra with many tracial states.*

EXAMPLE 2. Let K_n be the algebra of all $(a_n+b_n+2c_n)\times(a_n+b_n+2c_n)$ complex matrices and π_n be a representation of $\mathcal{S}(3)$ into K_n with

$$\pi_n = a_n\iota \oplus b_n sgn \oplus c_n\pi.$$

If we take $a_n = n$, $b_n = n-1$ and $c_n = 1$ for all $n \in \mathbb{N}$, then we have

$$\frac{1}{2n+1}Tr(\pi_n(g)) = \begin{cases} 1 \ , & \text{if } g=e \\ \dfrac{1}{2n+1}, & \text{if } g=(1,2),(1,3) \text{ or } (2,3) \\ \dfrac{2n-2}{2n+1}, & \text{if } g=(1,2,3) \text{ or } (1,3,2). \end{cases}$$

Hence the normal subgroup K of $G = \mathcal{S}(3)$ is trivial. On the other hand, since the left regular representation λ of $\mathcal{S}(3)$ is

$$\lambda = \iota \oplus sgn \oplus 2\pi \quad \text{and} \quad \pi\otimes\pi = \iota\oplus sgn\oplus\pi,$$

the tensor product representation $\bigotimes_{n=k}^{l}\pi_n$ are not any multiple of λ. Hence the fixed point algebra $\mathfrak{A}^\alpha$ is not a UHF-algebra but has a unique tracial state by Proposition 1 and 2.

PROPOSITION 4. *There exists a non-UHF but AF-algebra with a unique tracial state.*

EXAMPLE 3. In Example 2, take for $n \in \mathbb{N}$

$$a_n = b_n = n \quad \text{and} \quad c_n = \begin{cases} 1, & \text{if } n \text{ is even} \\ 2n, & \text{if } n \text{ is odd.} \end{cases}$$

Then, by an easy computation, we know that $\pi_{2k} \otimes \pi_{2k+1}$ is a $2(2k+1)^2$-multiple of λ (π_{2k} is not any multiple of λ) for $k \in \mathbb{N}$. Therefore this fixed point algebra is a UHF-algebra.

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