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SPECTRAL INCLUSIONS

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1. Introduction

In this paper, X is an abstract Banach space over complex numbers $\mathbb{C}$ and $\mathcal{L}(X)$ denotes the Banach algebra of all bounded linear operators defined on a Banach space X. Given an operator $T \in \mathcal{L}(X)$, let $Lat(T)$ stand for the collection of all closed linear subspaces of X which are invariant under T. T^* denotes the dual operator of $T \in \mathcal{L}(X)$.

An operator $T \in \mathcal{L}(X)$ is said to have the *Bishop's property (β)* if for every open subset U of $\mathbb{C}$ and for every sequence of analytic functions $f_n : U \to X$ for which $(T - \lambda)f_n(\lambda)$ converges uniformly in norm to zero on each compact subset of U, it follows that $f_n(\lambda) \to 0$ as $n \to \infty$, uniformly on each compact subset of U. Clearly, property (β) implies that T has the *single-valued extension property* (abbrev. SVEP) which means that for every open subset U of $\mathbb{C}$, the only analytic solution $f : U \to X$ of the equation $(T - \lambda)f(\lambda) = 0$ for all $\lambda \in U$ is the constant $f \equiv 0$.

An operator $T \in \mathcal{L}(X)$ is said to have the *decomposition property (δ)* if given an arbitrary open covering $\{U_1, U_2\}$ of $\mathbb{C}$, every $x \in X$ has a decomposition $x = u_1 + u_2$, where $u_1, u_2 \in X$ satisfy $u_j = (T - \lambda)f_j(\lambda)$ for all on $\mathbb{C} \setminus \overline{U_j}$ and some analytic function $f_j : \mathbb{C} \setminus \overline{U_j} \to X$ for $j = 1$, 2. An operator $T \in \mathcal{L}(X)$ is called *decomposable* if for every open cover $\{U_1, U_2\}$ of the complex plane $\mathbb{C}$ there exist $Y_1, Y_2 \in Lat(T)$ such that $Y_1 + Y_2 = X$ and $\sigma(T|Y_j) \subset U_j$ for $j = 1, 2$.

For various examples and characterizations of decomposable operators, see [3], [5].

If T has the SVEP, we define the *local resolvent set* of T at $x \in X$, denoted by $\rho_T(x)$, as the set of all $\lambda \in \mathbb{C}$ for which there exist an open neighborhood U of λ in $\mathbb{C}$ and an analytic function $f : U \to X$ with

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$(T-\mu)f(\mu) = x$ for all $\mu \in U$. The set $\sigma_T(x) := \mathbb{C} \setminus \rho_T(x)$ is called the *local spectrum* of T at the point $x \in X$.

Given an operator $T \in \mathcal{L}(X)$ and a subset A of $\mathbb{C}$, let $X_T(A) := \{ x \in X \mid \sigma_T(x) \subseteq A \}$. For each closed $F \subseteq \mathbb{C}$, let $\mathcal{X}_T(F)$ denote the space of all $x \in X$ for which there exists an analytic function $f : \mathbb{C} \setminus F \to X$ with $(T - \mu I)f(\mu) = x$ for all $\mu \in \mathbb{C} \setminus F$.

It is clear that if T has the SVEP, then $X_T(F) = \mathcal{X}_T(F)$ for every closed subset F of $\mathbb{C}$. In general, $X_T(F)$ is not necessarily closed linear subspace of X even if F is closed in $\mathbb{C}$, see [3].

An operator $T \in \mathcal{L}(X)$ is said to have the *Dunford's property (C)* if $X_T(F)$ is closed in norm for each closed subset F of $\mathbb{C}$.

We recall that, by [1], [14],
Bishop's property $(\beta) \Longrightarrow$ Dunford's property (C) $\Longrightarrow$ SVEP.

$$T \in \mathcal{L}(X) \text{ property } (\beta) \Longleftrightarrow T^* \in \mathcal{L}(X^*) \text{ property } (\delta).$$

$$T \in \mathcal{L}(X) \text{ decomposable} \Longleftrightarrow T \in \mathcal{L}(X) \text{ properties } (\beta),\ (\delta).$$

2. Properties preserved under asymptotic similarity

The spectrum of a linear operator T can be divided into subsets in many different ways, depending on the purpose of the inquiry. In [10], K.B. Laursen introduced the concept of a analytic residuum and then used this concept to study decomposability and single-valued extension property. It is shown in [3] that spectrum, local spectrum, analytic spectral subspace and single-valued extension property are preserved under quasi-nilpotent equivalent. In this paper, motivated by [3] we shall obtain theorems of this type for considerably more general classes.

Let $\sigma_p(T)$, $\sigma_{sur}(T)$ and $\sigma_{ap}(T)$ denote the *point spectrum, surjective spectrum* and *approximate point spectrum* of $T \in \mathcal{L}(X)$. Thus $\sigma_{sur}(T)$ consists of all $\lambda \in \mathbb{C}$ for which $T - \lambda$ is not surjective. What happens here is actually typical of the way in which the holes are filled in when we pass from $\sigma_{sur}(T)$ to $\sigma(T)$. Thus, if $\sigma_{sur}(T)$ and $\sigma(T)$ are different then $\sigma(T)$ is obtained from $\sigma_{sur}(T)$ by filling in the bounded components of the complement of $\sigma_{sur}(T)$ in $\sigma(T)$. To see this we need to introduce the analytic residuum.

DEFINITION 2.1. Let T be a bounded linear operator on a Banach space X. We define the ***analytic residuum***, denoted by $\mathcal{S}(T)$, as the set of all $\lambda \in \mathbb{C}$ for which for every neighborhood N_λ of λ there is a neighborhood $U \subseteq N_\lambda$ and a non-zero analytic function $f : U \to X$ satisfying $(T-\mu)f(\mu) = 0$ on U.

Note that T has the SVEP if and only if $\mathcal{S}(T) = \phi$.

THEOREM 2.2 [10]. *If T is a bounded linear operator on a Banach space X then $\sigma(T) = \mathcal{S}(T) \bigcup \sigma_{sur}(T)$.*

There are more interesting ways to express Theorem 2.2, if T has the SVEP, then T is invertible if and only if it is surjective.

The following properties of the surjective spectrum will be useful.

PROPOSITION 2.3 [10]. *If T is a bounded linear operator on a Banach space X, then $\sigma_{sur}(T)$ is compact with $\partial\sigma(T) \subseteq \sigma_{sur}(T) \subseteq \sigma(T) = \sigma_p(T) \bigcup \sigma_{sur}(T)$, where $\partial\sigma(T)$ denotes the boundary of $\sigma(T)$. Also,*

$$\sigma_{sur}(T) = \bigcup_{x \in X} \sigma_T(x), \quad \sigma_{sur}(T) = \sigma_{ap}(T^*) \quad \text{and} \quad \sigma(T^*) = \sigma_{ap}(T).$$

Finally, if T has the SVEP, then $\sigma_{sur}(T) = \sigma(T)$ and if T^* has the single-valued extension property, then $\sigma(T) = \sigma_{ap}(T)$.

Let X and Y be complex Banach spaces, and let $\mathcal{L}(X,Y)$ denote the space of all continuous linear operators from X to Y. For given operators $T \in \mathcal{L}(X)$ and $S \in \mathcal{L}(Y)$, we introduce the operator $C(S,T)$ on the Banach space $\mathcal{L}(X,Y)$ by

$$C(S,T)(A) := SA - AT$$

for $A \in \mathcal{L}(X,Y)$. Also, for all $n \in \mathbb{N}$ and all $A \in \mathcal{L}(X,Y)$, we have $C(S,T)^n A := C(S,T)^{n-1}(SA - AT) = \sum_{k=0}^{n} \binom{n}{k}(-1)^k S^{n-k} A T^k$.

An operator T defined on a Hilbert space H is said to be ***quasi-invertible*** if T has zero kernel and dense range. Operators $S \in \mathcal{L}(H)$ and $T \in \mathcal{L}(H)$ are ***quasi-similar*** if there are quasi-invertible operators $A, B \in \mathcal{L}(H)$ which satisfy $C(S,T)(A) = 0$ and $C(T,S)(B) = 0$.

In [6], [11] Sz. Nagy, C. Foias and T.B. Hoover show that quasi-similarity need not preserved the spectrum and compactness.

DEFINITION 2.4. An operator $A \in \mathcal{L}(X,Y)$ is said to *intertwine* S and T *asymptotically* if $\lim_{n\to\infty} \|C(S,T)^n(A)\|^{\frac{1}{n}} = 0$. Moreover, an operators $T \in \mathcal{L}(X)$ and $S \in \mathcal{L}(Y)$ are called the *asymptotically similar* if there exists $A \in \mathcal{L}(X,Y)$ such that A intertwines S and T asymptotically and its inverse A^{-1} intertwines T and S asymptotically.

In particular, we say that $T, S \in \mathcal{L}(X)$ are *quasi-nilpotent equivalent* if $\lim_{n\to\infty} \|C(S,T)^n(I)\|^{\frac{1}{n}} = \lim_{n\to\infty} \|C(T,S)^n(I)\|^{\frac{1}{n}} = 0$, and it is denoted by $T \overset{q}{\sim} S$.

THEOREM 2.5. *If $A \in \mathcal{L}(X,Y)$ intertwines S and T asymptotically, then the analytic residuum of T is contained in the analytic residuum of S.*

Proof. Let $\mu \in S(T)$ and let N_μ be an arbitrary neighborhood of μ. Then there is a neighborhood $U \subseteq N_\mu$ and a non-zero analytic function $f : U \longrightarrow X$ satisfying $(T-\lambda)f(\lambda) = 0$ on U. Consider a pair of concentric closed discs E and U such that $E \subset D \subset U$ with radii $0 < s < r$, and choose a constant $M > 0$ such that $\|f(\lambda)\| < M$ for all $\lambda \in D$. Then for each $\lambda \in E$, we obtain from Cauchy's integral formula that

$$(1) \quad \left\|\frac{f^{(n)}(\lambda)}{n!}\right\| = \left\|\frac{1}{2\pi i}\int_{\partial D}(z-\lambda)^{-n-1}f(z)\,dz\right\| \le Mr(r-s)^{-n-1},$$

for all $n = 0,1,2,\ldots.$ By the assumption, for $\epsilon := (r-s)/2$ there exists some constant $0 \le L$ such that $\|C(S,T)^n(A)\| \le L\epsilon^n$ for all $n = 0,1,2,\ldots$. An obvious combination of these estimates yields

$$(2) \qquad \left\|C(S,T)^n(A)\frac{f^{(n)}(\lambda)}{n!}\right\| \le MLr(r-s)^{-1}2^{-n},$$

for all $\lambda \in E$ and $n = 0,1,2,\ldots$. Now, consider the infinite series

$$g(\lambda) := \sum_{n=0}^{\infty}(-1)^n C(S,T)^n(A)\frac{f^{(n)}(\lambda)}{n!} \quad \text{for all } \lambda \in U.$$

It is clear from (1) and (2) that the infinite series defining $g(\lambda)$ converges uniformly on E and hence uniformly on each compact subset of U. From $(T-\lambda)f(\lambda)=0$ for all $\lambda \in U$, we obtain by induction that $(T-\lambda)f^{(n)}(\lambda) = nf^{(n-1)}(\lambda)$ for all $\lambda \in U$ and $n \in \mathbb{N}$. Since $SC(S,T)^n(A) = C(S,T)^{n+1}(A) + C(S,T)^n(A)T$ for all $n = 0,1,2,\dots$, we have

$$\begin{aligned}
&(S-\lambda)g(\lambda)\\
&= \sum_{n=0}^{\infty}(-1)^n(S-\lambda)C(S,T)^n(A)\frac{f^{(n)}(\lambda)}{n!}\\
&= \sum_{n=0}^{\infty}(-1)^n\Big(C(S,T)^{n+1}(A) + C(S,T)^n(A)(T-\lambda)\Big)\frac{f^{(n)}(\lambda)}{n!}\\
&= \sum_{n=1}^{\infty}(-1)^n\Big(C(S,T)^{n+1}(A)\frac{f^{(n)}(\lambda)}{n!} + C(S,T)^n(A)\frac{f^{(n-1)}(\lambda)}{(n-1)!}\Big)\\
&\qquad + C(S,T)(A)f(\lambda) + A(T-\lambda)f(\lambda).
\end{aligned}$$

Hence $(S-\lambda)g(\lambda) = A(T-\lambda)f(\lambda) = 0$, and so $\mu \in \mathcal{S}(S)$.

The following result is an immediate consequence of Theorem 2.5.

COROLLARY 2.6. *Assume that $A \in \mathcal{L}(X,Y)$ intertwines S and T asymptotically. If S has the single-valued extension property, then so is T.*

COROLLARY 2.7. *Analytic residuum is preserved under asymptotic similarity. In particular, if $T \overset{q}{\sim} S$, then $\mathcal{S}(T) = \mathcal{S}(S)$.*

Proof. Assume that T and S are asymptotically similar and choose a corresponding bijection $A \in \mathcal{L}(X,Y)$ for the asymptotic intertwining of (S,T) and (T,S). Then by Theorem 2.5, $\mathcal{S}(T) \subseteq \mathcal{S}(S)$. Since A^{-1} intertwines T and S asymptotically, it follows from Theorem 2.5 that $\mathcal{S}(S) \subseteq \mathcal{S}(T)$ and hence $\mathcal{S}(S) = \mathcal{S}(T)$.

COROLLARY 2.8 [3]. *Let $T, S \in \mathcal{L}(X)$. If T has the SVEP and $T \overset{q}{\sim} S$, then S has the SVEP.*

Proof. By Corollary 2.7, $\mathcal{S}(S) = \mathcal{S}(T) = \phi$.

LEMMA 2.9. *Assume that* $A \in \mathcal{L}(X,Y)$ *intertwines* S *and* T *asymptotically, then* $AX_T(F) \subseteq Y_S(F)$ *and* $A\mathcal{X}_T(F) \subseteq \mathcal{Y}_S(F)$ *for all closed subset* F *of* $\mathbb{C}$.

Proof. Let $x \in X$ and let $\mu \in \rho_T(x)$. Then there exists an open neighborhood U of μ in $\mathbb{C}$ and an analytic function $f : U \longrightarrow X$ on an open subset U of $\mathbb{C}$ such that $(T-\lambda)f(\lambda) = x$ for all $\lambda \in U$. It is easily checked that the series

$$h(\lambda) := \sum_{n=0}^{\infty} (-1)^n C(S,T)^n(A) \frac{f^{(n)}(\lambda)}{n!} \quad \text{for all } \lambda \in U$$

converges uniformly on each compact subset of U, and hence defines an analytic function $h : U \to Y$ such that $(S-\lambda)h(\lambda) = Ax$ for all $\lambda \in U$. Thus $\mu \in \rho_S(Ax)$ and hence $\sigma_S(Ax) \subseteq \sigma_T(x)$. This implies that $AX_T(F) \subseteq Y_S(F)$. A similar argument ensures that $A\mathcal{X}_T(F) \subseteq \mathcal{Y}_S(F)$ for all closed subset F of $\mathbb{C}$.

COROLLARY 2.10. *Assume that* T *and* S *are asymptotically similar. Then* $AX_T(F) = Y_S(F)$ *and* $A\mathcal{X}_T(F) = \mathcal{Y}_S(F)$ *for all closed subset* F *of* $\mathbb{C}$. *Moreover, if* $T, S \in \mathcal{L}(X)$ *are decomposable, then* $T \overset{q}{\sim} S$ *if and only if* $X_T(F) = X_S(F)$ *for every closed* $F \subseteq \mathbb{C}$.

COROLLARY 2.11. *If* $\sigma(T) \bigcap \sigma(S) = \phi$, *then the zero operator is the only operator* $A \in \mathcal{L}(X,Y)$ *which intertwines* S *and* T *asymptotically.*

Proof. $AX = A\mathcal{X}_T(\sigma(T)) \subseteq \mathcal{Y}_S(\sigma(T)) = \mathcal{Y}_S(\sigma(T) \bigcap \sigma(S)) = \{0\}$.

THEOREM 2.12. *Dunford's property* (C), *property* (δ), *Bishop's property* (β) *and decomposability are preserved under asymptotic similarity.*

Proof. Suppose that T and S are asymptotically similar and choose a corresponding bijection $A \in \mathcal{L}(X,Y)$ for the asymptotic intertwining of (S,T) and (T,S). By Corollary 2.10, we have $AX_T(F) = Y_S(F)$ and $A^{-1}Y_S(F) = X_T(F)$ for all closed subsets F of $\mathbb{C}$. This shows that Dunford's property (C) carries over from T to S. We also have $A\mathcal{X}_T(F) = \mathcal{Y}_S(F)$ and $A^{-1}\mathcal{Y}_S(F) = \mathcal{X}_T(F)$ for all closed subsets F of $\mathbb{C}$, which implies that property (δ) is preserved. Since property (δ) is preserved under asymptotic similarity and the properties (β) and (δ)

are dual to each other, property (β) is preserved by asymptotic similarity. Since both properties (C) and (δ) are preserved under asymptotic similarity, decomposability is also preserved.

COROLLARY 2.13. *Assume that $A \in \mathcal{L}(X,Y)$ intertwines S and T asymptotically. If A is onto, then $\sigma_{sur}(S) \subseteq \sigma_{sur}(T)$ and $\sigma(S) \subseteq \sigma(T)$.*

Proof. It follows from Proposition 2.3 and Lemma 2.9 that

$$\sigma_{sur}(S) = \bigcup_{y\in Y} \sigma_S(y) = \bigcup_{x\in X} \sigma_S(Ax) \subseteq \bigcup_{x\in X} \sigma_T(x) = \sigma_{sur}(T).$$

Finally, from Theorem 2.2 and Theorem 2.5, we conclude that

$$\sigma(T) = \mathcal{S}(T) \bigcup \sigma_{sur}(T) \subseteq \mathcal{S}(S) \bigcup \sigma_{sur}(S) = \sigma(S).$$

THEOREM 2.14. *Surjective spectrum, approximate point spectrum and spectrum are preserved under asymptotic similarity. In particular, if $T \overset{q}{\sim} S$, then $\sigma_{sur}(T) = \sigma_{sur}(S)$ and $\sigma(T) = \sigma(S)$.*

Proof. Assume that T and S are asymptotically similar and choose a corresponding bijection $A \in \mathcal{L}(X,Y)$ for the asymptotic intertwining of (S,T) and (T,S). By Corollary 2.13, $\sigma_{sur}(S) \subseteq \sigma_{sur}(T)$ and $\sigma(S) \subseteq \sigma(T)$. Since $A^{-1} \in \mathcal{L}(Y,X)$ intertwines T and S asymptotically and $\sigma_T(A^{-1}y) \subseteq \sigma_S(y)$ for any $y \in Y$, we have, by Proposition 2.3

$$\sigma_{sur}(T) = \bigcup_{x\in X} \sigma_T(x) = \bigcup_{y\in Y} \sigma_T(A^{-1}y) \subseteq \bigcup_{y\in Y} \sigma_S(y) = \sigma_{sur}(S).$$

From Theorem 2.2 and Corollary 2.7, we obtain

$$\sigma(T) = \mathcal{S}(T) \bigcup \sigma_{sur}(T) = \mathcal{S}(S) \bigcup \sigma_{sur}(S) = \sigma(S).$$

Finally, since S^* and T^* are asymptotically similar,

$$\sigma_{ap}(T) = \sigma_{sur}(T^*) = \sigma_{sur}(S^*) = \sigma_{ap}(S).$$

It is well known that if X is a Banach space and $T \in \mathcal{L}(X)$, then the boundary of $\sigma(T)$ is contained in $\sigma_{ap}(T)$.

COROLLARY 2.15. *Suppose that $T \in \mathcal{L}(X)$ and $S \in \mathcal{L}(Y)$ are asymptotically similar, then $\sigma_{sur}(T) \bigcap \sigma_{ap}(S) \neq \phi$. In particular, if $T \overset{q}{\sim} S$, then $\sigma_{sur}(T) \bigcap \sigma_{ap}(S) \neq \phi$.*

Proof. Suppose that $\sigma_{sur}(T) \bigcap \sigma_{ap}(S) = \phi$. Then $\sigma_{ap}(S) \subset \mathbb{C} \setminus \sigma_{sur}(T) = \mathbb{C} \setminus \sigma_{sur}(S)$ and so $\sigma_{sur}(S) \bigcap \sigma_{ap}(S) = \phi$. It follows from Proposition 2.3 that $\partial\sigma(S) \subseteq \sigma_{sur}(S) \bigcap \sigma_{ap}(S) = \phi$, and so $\partial\sigma(S) = \phi$. This contradiction shows that $\sigma_{sur}(T) \bigcap \sigma_{ap}(S) \neq \phi$.

COROLLARY 2.16. *Assume that $A \in \mathcal{L}(X, Y)$ intertwines S and T asymptotically. If A has dense range and S has the Dunford's property (C), then $\sigma(S) = \sigma_{sur}(S) \subseteq \sigma_{sur}(T)$.*

Proof. Since $\sigma_{sur}(T)$ is compact and $X = X_T(\sigma_{sur}(T))$, we have $Y = \overline{AX} = \overline{AX_T(\sigma_{sur}(T))} \subseteq \overline{Y_S(\sigma_{sur}(T))} = Y_S(\sigma_{sur}(T))$, which says that $\sigma_{sur}(S) = \sigma(S) = \sigma(S|Y_S(\sigma_{sur}(T))) \subseteq \sigma_{sur}(T)$.

PROPOSITION 2.17. *Suppose that $A \in \mathcal{L}(X, Y)$ intertwines $S \in \mathcal{L}(Y)$ and $T \in \mathcal{L}(X)$ asymptotically, then $\sigma_p(T) \subseteq \sigma_{ap}(S) \bigcap \sigma_{ap}(T)$.*

Proof. Clearly, $\sigma_p(T) \subseteq \sigma_{ap}(T)$. If $\lambda \in \sigma_p(T)$, then there exists a $x(\neq 0) \in X$ such that $(T - \lambda)x = 0$. Suppose that $\lambda \notin \sigma_{ap}(S) \bigcap \sigma_{ap}(T)$. Then there exists a constant $m > 0$ such that $m\|y\| \leq \|(S - \lambda)y\|$ for all $y \in Y$. Since $(T - \lambda)^n x = 0$ for all $n \in \mathbb{N}$,

$$\begin{aligned} C(S,T)^n(A)x &= C(S - \lambda, T - \lambda)^n(A)x \\ &= \sum_{k=0}^{n} \binom{n}{k} (-1)^k (S - \lambda)^{n-k} A (T - \lambda)^k x \\ &= (S - \lambda)^n Ax. \end{aligned}$$

Therefore $m\|Ax\|^{\frac{1}{n}} \leq \|C(S,T)^n(A)x\|^{\frac{1}{n}}$ for all $n \in \mathbb{N}$. Since A intertwines S and T asymptotically and since $Ax \neq 0$ by the injectivity of A, we conclude that $m = 0$. This contradiction shows that $\sigma_p(T) \subseteq \sigma_{ap}(S) \bigcap \sigma_{ap}(T)$.

We recall, by [9] that the operator T has *finite ascent* if for every $\lambda \in \mathbb{C}$ there is an $n \in \mathbb{N}$ such that $Ker(T - \lambda)^n = Ker(T - \lambda)^{n+1}$.

PROPOSITION 2.18. *Point spectrum and finite ascent are preserved under quasi-similarity.*

Proof. Assume that $S \in \mathcal{L}(X)$ and $T \in \mathcal{L}(X)$ are quasi-similar. Then there exists quasi-invertible operators $A, B \in \mathcal{L}(X)$ such that $C(S,T)(A) = 0$ and $C(T,S)(B) = 0$. If $x \in X$ is an eigenvector for the eigenvalue λ of T, then $(S-\lambda)Ax = A(T-\lambda)x = 0$ and hence $\lambda \in \sigma_p(S)$ by the injectivity of A. Thus $\sigma_p(T) \subseteq \sigma_p(S)$. By the same reasoning, we conclude that $\sigma_p(S) \subseteq \sigma_p(T)$.

Assume that $Ker(S-\lambda)^n = Ker(S-\lambda)^{n+1}$ for some $n \in \mathbb{N}$. If $x \in Ker(T-\lambda)^{n+1}$, then $(S-\lambda)^{n+1}Ax = A(T-\lambda)^{n+1}x = 0$, and so $Ax \in Ker(S-\lambda)^{n+1} = Ker(S-\lambda)^n$. Thus $0 = (S-\lambda)^n Ax = A(T-\lambda)^n x$ and hence $(T-\lambda)^n x = 0$ by the injectivity of A. Hence T has finite ascent.

COROLLARY 2.19. *Assume that $A \in \mathcal{L}(X)$ has dense range and intertwines S and T in the sense that $SA = AT$. If T^* has finite ascent, then S^* has finite ascent.*

Proof. Since A has dense range and $SA = AT$, A^* is injective and $T^*A^* = A^*S^*$. Whence the Proposition 2.18 is applied.

An operator T on a Banach space X is said to be ***totally paranormal*** if $\|(T-\lambda)x\|^2 \leq \|(T-\lambda)^2x\|\|x\|$ for all $x \in X$ and for every $\lambda \in \mathbb{C}$.

As noted in [9], the totally paranormal operators form a proper subclass of paranormal operators. It is easily check that every totally paranormal operator T has finite ascent.

COROLLARY 2.20. *Let $T \in \mathcal{L}(X)$ be a totally paranormal and let F be a closed subset of $\mathbb{C}$. If $T \in \mathcal{L}(X)$ and $S \in \mathcal{L}(Y)$ are asymptotically similar, then $X_S(F)$ is closed.*

Proof. It follows from ([9], Proposition 4.14) that $X_T(F)$ is closed. Hence $X_S(F)$ is closed.

COROLLARY 2.21. *Let $T \in \mathcal{L}(H)$ be a totally paranormal operator on the Hilbert space H with $\sigma_p(T) = \phi$ and let $F \subseteq \mathbb{C}$ be closed. If $T \overset{q}{\sim} S$, then $H_S(F) = \bigcap_{\lambda \notin F}(T-\lambda)H$.*

Proof. It is clear from Theorem 2.14 and ([9], Theorem 4.15) that $H_S(F) = H_T(F) = \bigcap_{\lambda \notin F}(T-\lambda)H$.

Let X_i and Y_i be complex Banach spaces, and let $\mathcal{L}(X_1 \oplus X_2, Y_1 \oplus Y_2)$ denote the space of all continuous linear operators from $X_1 \oplus X_2$ to $Y_1 \oplus Y_2$. Given operators $T_i \in \mathcal{L}(X_i)$ and $S_i \in \mathcal{L}(Y_i)$, we introduce the operator $C(S_1 \oplus S_2, T_1 \oplus T_2)$ on the space $\mathcal{L}(X_1 \oplus X_2, Y_1 \oplus Y_2)$ by $C(S_1 \oplus S_2, T_1 \oplus T_2)(A \oplus B) := (S_1 \oplus S_2)(A \oplus B) - (A \oplus B)(T_1 \oplus T_2)$ for $A \oplus B \in \mathcal{L}(X_1 \oplus X_2, Y_1 \oplus Y_2)$.

LEMMA 2.22. *Let $T_i \in \mathcal{L}(X_i)$, $S_i \in \mathcal{L}(Y_i)$ and $A_i \in \mathcal{L}(X_i, Y_i)$. If $A_1 \oplus A_2 \in \mathcal{L}(X_1 \oplus X_2, Y_1 \oplus Y_2)$ are intertwines $S_1 \oplus S_2$ and $T_1 \oplus T_2$ asymptotically, then A_i are intertwines T_i and S_i asymptotically for $i = 1, 2$.*

Proof. For every $n = 1, 2, \dots$, we have

$$\begin{aligned}
&C(S_1 \oplus S_2, T_1 \oplus T_2)^n(A_1 \oplus A_2) \\
&= \sum_{k=0}^{n} \binom{n}{k} (-1)^k (S_1 \oplus S_2)^{n-k}(A_1 \oplus A_2)(T_1 \oplus T_2)^k \\
&= \sum_{k=0}^{n} \binom{n}{k} (-1)^k S_1^{n-k} A_1 T_1^k \oplus \sum_{k=0}^{n} \binom{n}{k} (-1)^k S_2^{n-k} A_2 T_2^k \\
&= C(S_1, T_1)^n A_1 \oplus C(S_2, T_2)^n A_2.
\end{aligned}$$

Since

$$\|C(S_1 \oplus S_2, T_1 \oplus T_2)^n(A_1 \oplus A_2)\|^2 = \|C(S_1, T_1)^n A_1\|^2 + \|C(S_2, T_2)^n A_2\|^2$$

for every $n \in \mathbb{N}$, we obtain

$$\lim_{n \to \infty} \|C(S_i, T_i)^n A_i\|^{\frac{1}{n}} \le \lim_{n \to \infty} \|C(S_1 \oplus S_2, T_1 \oplus T_2)^n(A_1 \oplus A_2)\|^{\frac{1}{n}} = 0.$$

Hence A_i are intertwines T_i and S_i asymptotically for $i = 1, 2$.

COROLLARY 2.23. *Assume that $T_1 \oplus T_2 \in \mathcal{L}(X_1 \oplus X_2)$ and $S_1 \oplus S_2 \in \mathcal{L}(Y_1 \oplus Y_2)$ are asymptotically similar. If T_1 and T_2 are decomposable operators, then S_1 and S_2 are decomposable. Consequently, $S_1 \oplus S_2$ is decomposable.*

Proof. By Lemma 2.22, T_i and S_i are asymptotically similar. Since decomposability is preserved under asymptotically similar, S_1 and S_2 are decomposable.

PROPOSITION 2.24. *Let $T_i \in \mathcal{L}(X_i)$ $(i = 1,2)$. Then $\mathcal{S}(T_1 \oplus T_2) = \mathcal{S}(T_1) \bigcup \mathcal{S}(T_2)$.*

Proof. Let $\mu \in \mathcal{S}(T_1 \oplus T_2)$ and N_μ be an arbitrary neighborhood of μ. Then there is a neighborhood $U \subseteq N_\mu$ and a non-zero analytic function $f : U \longrightarrow X_1 \oplus X_2$ satisfying $(T_1 \oplus T_2 - \lambda)f(\lambda) = 0$ on U. Consider the map

$$f_j(\lambda) := P_j \circ f(\lambda),$$

where $P_j : X_1 \oplus X_2 \longrightarrow X_j$ $(j = 1,2)$ is the j^{th} projection operator. Then either f_1 or f_2 is a non-zero analytic function and $f = f_1 \oplus f_2$. Since $(T_1 - \lambda)f_1(\lambda) \oplus (T_2 - \lambda)f_2(\lambda) = 0$, we have $(T_j - \lambda)f_j(\lambda) = 0,\ j = 1,2$, and so $\mu \in \mathcal{S}(T_1) \bigcup \mathcal{S}(T_2)$. Hence $\mathcal{S}(T_1 \oplus T_2) \subseteq \mathcal{S}(T_1) \bigcup \mathcal{S}(T_2)$. To obtain the opposite inclusion, let $\mu \in \mathcal{S}(T_1) \bigcup \mathcal{S}(T_2)$, and let N_μ be an arbitrary neighborhood of μ. Then either there exist a neighborhood $U \subseteq N_\mu$ and a non-zero analytic function $g : U \to X_1$ satisfying $(T_1 - \lambda)g(\lambda) = 0$ on U or there exist a neighborhood $V \subseteq N_\mu$ and a non-zero analytic function $h : V \to X_2$ satisfying $(T_2 - \lambda)h(\lambda) = 0$ on U. We may assume that there exist a neighborhood $W \subseteq N_\mu$ and a non-zero analytic function $f : W \to X_1 \oplus X_2$ satisfying $(T_1 \oplus T_2 - \lambda)f(\lambda) = 0$ on W. Thus $\mu \in \mathcal{S}(T_1 \oplus T_2)$. Hence $\mathcal{S}(T_1 \oplus T_2) = \mathcal{S}(T_1) \bigcup \mathcal{S}(T_2)$.

COROLLARY 2.25. *$T := T_1 \oplus T_2 \in \mathcal{L}(X_1 \oplus X_2)$ has the single valued extension property if and only if T_1 and T_2 have this property. In this case, $\sigma_T(x_1 \oplus x_2) = \sigma_{T_1}(x_1) \bigcup \sigma_{T_2}(x_2)$.*

Proof. Since $\mathcal{S}(T_1 \oplus T_2) = \mathcal{S}(T_1) \bigcup \mathcal{S}(T_2) = \phi$, one has the result.

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