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HOMOCLINIC ORBITS FOR SECOND ORDER HAMILTONIAN SYSTEMS

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0. Introduction

Let us consider the following second order Hamiltonian systems of the form;

$$\text{(HS)} \qquad \ddot{q} - L(t)q + V_q(t,q) = 0, \qquad q \in \mathbf{R}^n.$$

We assume that the $n \times n$ matrix $L(t)$ satisfies

$$\text{(L)} \qquad L(t) \in C(\mathbf{R},\ \mathbf{R}^{n^2}),$$

is T-periodic in t, and is symmetric and positive definite uniformly for $t \in [0,T]$. The function V satisfies

(V_1) $V \in C^2(\mathbf{R} \times \mathbf{R}^n,\ \mathbf{R})$ and $V(t,q)$ is T-periodic in t,
(V_2) $V_{qq}(t,0) = 0$,
(V_3) There is a $\mu > 2$ such that

$$0 < \mu V(t,q) \le \langle q, V_q(t,q)\rangle \text{ for all } q \in \mathbf{R}^n \setminus \{0\},$$

V. Coti Zelati and Paul H. Rabinowitz [**3**] proved the existence of infinitely many homoclinic solutions for the problem (HS) under the conditions (V_1), (V_2), (V_3), and $(*)$. Here $(*)$ is the condition that there exist only finite number of critical points of the corresponding functional I of the problem (HS) whose critical values are less than a certain number and will be explained later. They have even suggested that the condition $(*)$ could be replaced with the weaker condition $(**)$ in [**3**] ,which asserts the discreteness of the critical values instead of the finiteness of the critical values as in $(*)$ and will be explained below. Moreover they have

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shown that the same results could be obtained under a weaker condition (**) if we further assume that V satisfies one more condition

(V_4) For all $\xi \in S^{n-1}, s \mapsto \frac{1}{s}\langle \xi, V_q(t, s\xi)\rangle$ is an increasing function of s.

Therefore it is natural to ask with what kind of potentials the corresponding functionals satisfy (**). In this paper we give the condition on V whose corresponding functional satisfies (**) and so (HS) with this potential has infinitely many homoclinic solutions.

1. Preliminaries

Let $E = W^{1,2}(\mathbf{R}, \mathbf{R}^n)$ under the usual norm

$$\left(\int_{-\infty}^{\infty} (|\dot{q}|^2 + |q|^2) dt\right)^{1/2}.$$

Thus E is a Hilbert space and $E \subset C^0(\mathbf{R}, \mathbf{R}^n)$, the space of continuous function q on $\mathbf{R}$ such that $q(t) \to 0$ as $|t| \to \infty$. We will seek solutions of (HS) as critical points of the functional I associated with (HS) and given by

$$I(q) = \frac{1}{2}\int_{-\infty}^{\infty} (|\dot{q}|^2 + \langle q, L(t)q\rangle) dt - \int_{-\infty}^{\infty} V(t, q) dt.$$

By (L),

$$\|q\|^2 = \int_{-\infty}^{\infty} (|\dot{q}|^2 + \langle q, L(t)q\rangle) dt$$

can and will be taken as an equivalent norm on E. If $q \in E$, $j \in \mathbf{Z}$, and $\tau_j(q) = q(t - jT)$, then $I(\tau_j q) = I(q)$. Hence I possesses a $\mathbf{Z}$-action. It is standard that the critical points of I in E correspond to the homoclinic solutions for (HS). However to apply the standard variational methods it is necessary that I satisfy the Palais-Smale condition which is abbreviated as the (PS) condition. But our functional does not satisfy the (PS)-condition.

LEMMA 1.1. *I does not satisfies the* (PS) *condition.*

Proof. Suppose $b > 0$ is a critical value of I with corresponding critical point q. Let $u_m = q + \tau_m q$. Then

$$\begin{aligned} I(u_m) &= \frac{1}{2}\|q + \tau_m q\|^2 - \int_{-\infty}^{\infty} V(t, q + \tau_m q)dt \\ &= \frac{1}{2}\|q + \tau_m q\|^2 - \int_{-\infty}^{\infty} V(t, q)dt - \int_{-\infty}^{\infty} V(t, \tau_m q)dt \\ &\quad - \int_{-\infty}^{\infty} (V(t, q + \tau_m q) - V(t, q) - V(t, \tau_m q))dt. \end{aligned}$$

Observe that $V(t, q + \tau_m q) - V(t, q) - V(t, \tau_m q) \to 0$ uniformly on $[-R, R]$, $0 < R < +\infty$. Choose $\delta > 0$ so that

$$|x| < \delta \quad \text{implies that} \quad |V_q(t, x)| \leq |x|.$$

Choose $M > 0$ so that

$$\begin{aligned} \|q + \tau_m q\|_{L^\infty} &\leq \sqrt{2}\|q + \tau_m q\| \\ &\leq 2\sqrt{2}\|q\| \\ &< M < +\infty. \end{aligned}$$

Let $M_1 = \sup_{\delta \leq |\xi| \leq M} |V_q(t, \xi)|$. Then we have

$$|V_q(t, x)| \leq (1 + \frac{M_1}{\delta})|x| \quad \text{for} \quad |x| \leq M.$$

Thus

$$\begin{aligned} |V(t, q + \tau_m q) - V(t, \tau_m q)| &\leq \langle q, V_q(t, \theta q + \tau_m q)\rangle \\ &\leq (1 + \frac{M_1}{\delta})|q|(|q| + |\tau_m q|), \ 0 < \theta < 1. \end{aligned}$$

Hence

$$\int_{|t|>R} |V(t, q + \tau_m q) - V(t, \tau_m q)|dt$$

$$= \int_{|t|>R} |\langle q, V_q(t, \theta q + \tau_m q)\rangle| dt$$
$$\leq (1 + \frac{M_1}{\delta})(\int_{|t|>R} |q|^2 dt)^{\frac{1}{2}} (\int_{|t|>R} (|q|^2 + |\tau_m q|^2) dt)^{1/2}$$
$$\leq 2\|q\|(1 + \frac{M_1}{\delta})(\int_{|t|>R} |q|^2 dt)^{1/2}.$$

Given $\varepsilon > 0$, we can choose R sufficiently large so that

$$\begin{aligned} |-\int_{|t|>R} V(t,q)dt| &= \int_{|t|>R} V(t,q)dt \\ &\leq \varepsilon \int_{|t|>R} |q|^2 dt \end{aligned}$$

and

$$\int_{|t|>R} |q|^2 dt < \varepsilon.$$

Therefore $\int_{-\infty}^{\infty}(V(t, q + \tau_m q) - V(t,q) - V(t, \tau_m q))dt \to 0$ as $m \to \infty$. Note also that $\|q + \tau_m q\|^2 = \|q\|^2 + \|\tau_m q\|^2 + \varepsilon_m$, $\varepsilon_m \to 0$ as $m \to +\infty$. Thus $I(u_m) \to 2b$. Let us now check $I'(u_m)$. For each $\varphi \in E$, we have

$$\begin{aligned} \langle I'(u_m), \varphi\rangle &= \langle I'(q + \tau_m q), \varphi\rangle \\ &= \int_{-\infty}^{\infty} (\langle \dot{q} + \tau_m \dot{q}, \dot{\varphi}\rangle + \langle \varphi, L(t)(q + \tau_m q)\rangle) dt \\ &\quad - \int_{-\infty}^{\infty} \langle \varphi, V_q(t, q + \tau_m q)\rangle dt \\ &= -\int_{-\infty}^{\infty} \langle \varphi, V_q(t, q + \tau_m q) - V_q(t, \tau_m q) - V_q(t,q)\rangle dt. \end{aligned}$$

Hence $I'(u_m) \to 0$. However

$$\begin{aligned} \|u_m - u_n\| &= \|\tau_m q - \tau_n q\| \\ &= \|q - \tau_{n-m} q\| \\ &= 2\|q\| + \varepsilon_{|n-m|}, \end{aligned}$$

where $\varepsilon_{|n-m|} \to 0$ as $|n-m| \to +\infty$. Therefore (u_m) has no convergent subsequence.

Given $q \in E \setminus \{0\}$, define a function $f : (0,\infty) \to \mathbf{R}$ by

$$\begin{aligned} f(s) &= I(sq) \\ &= \frac{s^2}{2}\int_{-\infty}^{\infty} (|\dot{q}|^2 + \langle q, L(t)q\rangle)dt - \int_{-\infty}^{\infty} V(t,sq)dt. \end{aligned}$$

Then

$$\begin{aligned} f'(s) &= s\int_{-\infty}^{\infty} (|\dot{q}|^2 + \langle q, L(t)q\rangle)dt - \int_{-\infty}^{\infty} \langle q, V_q(t,sq)\rangle dt \\ &= s\left(\int_{-\infty}^{\infty} (|\dot{q}|^2 + \langle q, L(t)q\rangle)dt - \frac{1}{s}\int_{-\infty}^{\infty} \langle q, V_q(t,sq)\rangle dt\right). \end{aligned}$$

Now (V_4) implies that $f : (0,\infty) \to \mathbf{R}$ has a unique maximum point. Moreover (V_1)-(V_3) implies that

$$V(t,x)\begin{cases} \le M|x|^\mu & \text{uniformly in } t \text{ for} \quad |x| \le 1, \\ \ge m|x|^\mu & \text{uniformly in } t \text{ for} \quad |x| \ge 1. \end{cases}$$

Here

$$\begin{aligned} m &= \min_{\substack{t\in\mathbf{R} \\ |x|=1}} V(t,x) > 0 \quad \text{and} \\ M &= \max_{\substack{t\in\mathbf{R} \\ |x|=1}} V(t,x) > 0. \end{aligned}$$

Hence $f(s) \to -\infty$ as $s \to +\infty$. Observe also that $I(q) = \frac{1}{2}\|q\|^2 + o(\|q\|^2)$. Therefore 0 is an isolated singular point of I. Choose a point $e \neq 0$ such that $I(e) \le 0$. Let

$$c = \inf_{g\in\Gamma_e} \max_{\theta\in[0,1]} I(g(\theta)),$$

where

$$\Gamma_e = \{g \in C([0,1],E) : g(0) = 0,\ g(1) = e\}.$$

Since $I(q) = \frac{1}{2}\|q\|^2 + o(\|q\|^2), \quad c > 0.$

From now on we use the following notations;

$$\begin{aligned} I^s &= \{q \in E | I(q) \leq s\}, \quad I_s = \{q \in E | I(q) \geq s\}, \\ I_a^b &= I_a \cap I^b, \quad \mathcal{K} = \text{the set of critical points of } I \\ \mathcal{K}_a^b &= \mathcal{K} \cap I_a^b. \end{aligned}$$

Recall that the key roles (PS) plays in the proof of the standard Deformation Theorem is that it provides a $\delta > 0$ such that $\|I'(x)\| \geq \delta$ for all $x \in I_{b-\varepsilon}^{b+\varepsilon}$ for some $\varepsilon > 0$ if $\mathcal{K}(b) \equiv \mathcal{K}_b^b = \emptyset$ and an appropriately modified statement if $\mathcal{K}(b) \neq \emptyset$. Since our functional I does not satisfy the (PS)-condition, we cannot use the standard Deformation Theorem in its naive form. However V. Coti Zelati and Paul H. Rabinowitz [3] escaped from this difficulty by imposing the condition

(*) there is an $\alpha > 0$ such that $I^{c+\alpha}/\mathbf{Z}$ contains only finitely many critical points of I.

Usually the value of c depends on the choice of e. But we have the following

LEMMA 1.2. *If V satisfies (V_1)–(V_3), then c is independent of the choice of e.*

Proof. Define a function $f : (0, \infty) \to \mathbf{R}$ by

$$\begin{aligned} f(s) &= I(sq) \\ &= \frac{s^2}{2} \int_{-\infty}^{\infty} (|\dot{q}|^2 + \langle q, L(t)q \rangle) dt - \int_{-\infty}^{\infty} V(t, sq) dt. \end{aligned}$$

Then

$$\begin{aligned} f'(s) &= s \int_{-\infty}^{\infty} (|\dot{q}|^2 + \langle q, L(t)q \rangle) dt - \int_{-\infty}^{\infty} \langle q, V_q(t, sq) \rangle dt \\ &\leq s \int_{-\infty}^{\infty} (|\dot{q}|^2 + \langle q, L(t)q \rangle) dt - \frac{\mu}{s} \int_{-\infty}^{\infty} V(t, sq) dt \end{aligned}$$

$$
\begin{aligned}
&= \frac{\mu}{s}\left(\frac{s^2}{\mu}\int_{-\infty}^{\infty}(|\dot q|^2 + \langle q, L(t)q\rangle)dt - \int_{-\infty}^{\infty} V(t, sq)dt\right) \\
&\le \frac{\mu}{s}\left(\frac{s^2}{2}\int_{-\infty}^{\infty}(|\dot q|^2 + \langle q, L(t)q\rangle)dt - \int_{-\infty}^{\infty} V(t, sq)dt\right) \\
&= \frac{\mu}{s} f(s).
\end{aligned}
$$

Hence we obtain $f'(s) - \mu/sf(s) \le 0$. This implies that $f(s)/s^{\mu}$ is a decreasing function of s. Therefore any two points $e_1 \neq 0$ and $e_2 \neq 0$ such that $e_1 \in I^0$ and $e_2 \in I^0$ can be joined by a path lying in I^0. This proves that c is independent of the choice e.

To define an another intrinsic constant $\bar{c}$, we need the following

LEMMA 1.3. *If $q \in \mathcal{K}$, then $I(q) \ge (\frac{1}{2} - \frac{1}{\mu})\|q\|^2$.*

Proof.

$$
\begin{aligned}
I(q) &= \frac{1}{2}\int_{-\infty}^{\infty}(|\dot q|^2 + \langle q, L(t)q\rangle)dt - \int_{-\infty}^{\infty} V(t, q)dt \\
\langle I'(q), q\rangle &= \int_{-\infty}^{\infty}(|\dot q|^2 + \langle q, L(t)q\rangle)dt - \int_{-\infty}^{\infty}\langle q, V_q(t, q)\rangle dt \\
&= 0.
\end{aligned}
$$

Hence

$$
\begin{aligned}
I(q) &= I(q) - \frac{1}{2}\langle I'(q), q\rangle \\
&= \int_{-\infty}^{\infty}\left(\frac{1}{2}\langle q, V_q(t, q)\rangle - V(t, q)\right)dt \\
&\ge \left(\frac{1}{2} - \frac{1}{\mu}\right)\int_{-\infty}^{\infty}\langle q, V_q(t, q)\rangle dt \\
&= \left(\frac{1}{2} - \frac{1}{\mu}\right)\int_{-\infty}^{\infty}(|\dot q|^2 + \langle q, L(t)q\rangle)dt \\
&= \left(\frac{1}{2} - \frac{1}{\mu}\right)\|q\|^2.
\end{aligned}
$$

Let

$$\bar{c} = \inf_{q \in \mathcal{K} \setminus \{0\}} I(q).$$

Since 0 is an isolated singular point, Lemma 1.3 implies that $\bar{c} > 0$. We now have two constants c and $\bar{c}$. To compare the two numbers c and $\bar{c}$, we need the following two Lemmas.

LEMMA 1.4 ([4]). *Let K be a compact metric space, $K_0 \subset K$ a closed set, X a Banach space, $\chi \in C(K_0, X)$ and let us define a complete metric space*

$$M = \{g \in C(K, X); g(s) = \chi(s) \text{ if } s \in K_0\}$$

with the usual distance d. Let $\varphi \in C^1(X, \mathbf{R})$ and let us define

$$c = \inf_{g \in M} \max_{s \in K} \varphi(g(s)).$$

Then for each sequence (f_k) in M such that

$$\max_K \varphi(f_k) \to c,$$

there exists a sequence (v_k) in X such that

$$\begin{aligned} &\varphi(v_k) \to c, \\ &dist(v_k, f_k(K)) \to 0, \\ &|\varphi'(v_k)| \to 0 \quad \text{as} \quad k \to +\infty. \end{aligned}$$

LEMMA 1.5 ([3]). *Let $(u_m) \subset E$ be such that $I(u_m) \to b > 0$ and $I'(u_m) \to 0$. Then there is an $\ell \in \mathbf{N}$ with ℓ bounded above by a constant depending only on b, normalized functions $v_1, v_2, \ldots, v_\ell \in \mathcal{K} \setminus \{0\}$, a subsequence of (u_m), and corresponding $(k_m^i) \subset \mathbf{Z}$, $1 \le i \le \ell$, such that*

$$\begin{aligned} &\|u_m - \sum_1^\ell \tau_{k_m^i} v_i\| \to 0, \\ &\sum_1^\ell I(v_i) = b, \end{aligned}$$

and, for $i \neq j$,

$$|k_m^i - k_m^j| \to +\infty$$

as $m \to \infty$ *along the subsequence.*

In the above Lemma we say that a function v is normalized if

$$\|v\|_{L^\infty} = \max_{t \in \mathbf{R}} |v(t)|$$

occurs for $t \in [0, T]$ and $|v(t)| < \|v\|_{L^\infty}$ for $t < 0$. We are now ready to show that $c = \bar{c}$.

THEOREM 1.1. *If* V *satisfies the conditions* (V_1)–(V_4), *then* $c = \bar{c}$.

Proof. Suppose $c < \bar{c}$. By Lemma 1.4 there exists a sequence $(u_m) \subset E$ such that $I(u_m) \to c$ and $I'(u_m) \to 0$. Since $c > 0$, we can apply Lemma 1.5 to obtain a normalized critical points $v_1, v_2, \ldots, v_\ell$ such that

$$\sum_{i=1}^{\ell} I(v_i) = c.$$

But this contradicts the fact that $\bar{c} = \inf_{q \in \mathcal{K} \setminus \{0\}} I(q)$. Therefore $c \geq \bar{c}$. On the other hand, given any $q \in \mathcal{K} \setminus \{0\}$, consider

$$\begin{aligned} f(s) &= I(sq) \\ &= \frac{s^2}{2} \int_{-\infty}^{\infty} (|\dot{q}|^2 + \langle q, L(t)q \rangle) dt - \int_{-\infty}^{\infty} V(t, sq) dt. \end{aligned}$$

Observe that

$$f'(s) = s\Big(\int_{-\infty}^{\infty} (|\dot{q}|^2 + \langle q, L(t)q \rangle) dt - \frac{1}{s} \int_{-\infty}^{\infty} \langle q, V_q(t, sq) \rangle dt\Big).$$

Since $q \in \mathcal{K} \setminus \{0\}$, $f'(1) = 0$. Now (V_4) implies that f attains its maximum value at $s = 1$. Therefore $c \leq f(1) = I(q)$ for any $q \in \mathcal{K} \setminus \{0\}$. Hence $c \leq \bar{c}$.

2. Homoclinic solutions

In this section we discuss the existence of infinitely many solutions of (HS). Using the fact that $c = \bar{c}$, we can show that c is a critical value of I, though I does not satisfy the (PS) condition.

THEOREM 2.1. *If V satisfies the conditions* (V_1)–(V_4), *then c is a critical value of I.*

Proof. Choose a sequence $(q_m) \subset \mathcal{K} \setminus \{0\}$ such that $I(q_m) \to \bar{c} = c$. Since $I(q) \geq (\frac{1}{2} - \frac{1}{\mu})\|q\|^2$ for all $q \in \mathcal{K}$, (q_m) is bounded in E. Hence there exists a subsequence (q_{m_j}) of (q_m) and $q \in E$ such that $q_{m_j} \rightharpoonup q$ in E. We may also assume that (q_m) is a normalized sequence. By Sobolev imbedding theorem we have $q_{m_j} \to q$ in $L^{loc}_{\infty}(\mathbf{R}, \mathbf{R}^n)$. Hence $q \neq 0$. Now

$$\begin{aligned} 0 = \langle I'(q_{m_j}), \varphi \rangle &= \int_{-\infty}^{\infty} (\langle \dot{q}_{m_j}, \dot{\varphi} \rangle + \langle \varphi, L(t) q_{m_j} \rangle) dt \\ &\quad - \int_{-\infty}^{\infty} \langle \varphi, V_q(t, q_{m_j}) \rangle dt. \end{aligned}$$

By taking limits we obtain

$$\begin{aligned} 0 &= \int_{-\infty}^{\infty} (\langle \dot{q}, \dot{\varphi} \rangle + \langle \varphi, L(t) q \rangle) dt - \int_{-\infty}^{\infty} \langle \varphi, V_q(t, q) \rangle dt \\ &= \langle I'(q), \varphi \rangle. \end{aligned}$$

Hence q is a critical point of I. Let $w_m = q_{m_j} - q$. Then as in Proposition 1.2 in [**3**] we can show that

$$\begin{aligned} &I(w_m) \to c - I(q), \\ &I'(w_m) \to 0. \end{aligned}$$

Now

$$I(w_m) = \frac{1}{2} \int_{-\infty}^{\infty} (|\dot{w}_m|^2 + \langle w_m, L(t) w_m \rangle) dt - \int_{-\infty}^{\infty} V(t, w_m) dt$$

and

$$\begin{aligned} \langle I'(w_m), w_m \rangle &= \int_{-\infty}^{\infty} (|\dot{w}_m|^2 + \langle w_m, L(t) w_m \rangle) dt \\ &\quad - \int_{-\infty}^{\infty} \langle w_m, V_q(t, w_m) \rangle dt. \end{aligned}$$

Hence

$$I(w_m) - \frac{1}{2}\langle I'(w_m), w_m\rangle \geq (\frac{\mu}{2} - 1)\int_{-\infty}^{\infty} V(t, w_m)dt$$
$$\geq 0.$$

Thus

$$0 \leq I(w_m) - \frac{1}{2}\langle I'(w_m), w_m\rangle \leq I(w_m) + M\|I'(w_m)\|$$

for some constant M independent of m. Therefore

$$0 \leq c - I(q).$$

Since $c = \overline{c} = \inf_{q\in\mathcal{K}\setminus\{0\}} I(q)$, this completes the proof.

The following fact is crucial to the existence of infinitely many homoclinic solutions of (HS).

LEMMA 2.1. *Let $q \in E$ be a critical point of I with $I(q) = c$. Choose $\overline{q}$ on the ray passing through 0 and q such that $I(\overline{q}) < 0$. Define a function $g : [0,1] \to E$ by $g(\theta) = \theta\overline{q}$. Then*

(1) $g \in \Gamma$,
(2) $\max_{\theta\in[0,1]} I(g(\theta)) = c$, *and*
(3) *for each $r > 0$, there exists $\varepsilon > 0$ such that $I(g(\theta)) > c-\varepsilon$ implies $g(\theta) \in B_r(q)$.*

Proof. (1) and (2) are evident from the construction of g and (V_4). Suppose $q = \overline{\theta}\overline{q}$, $0 < \overline{\theta} < 1$. Then for any $\varepsilon > 0$, by (V_4), there are constants $\theta_{-\varepsilon}$ and $\theta_{+\varepsilon}$ with $\theta_{-\varepsilon} < \overline{\theta} < \theta_{+\varepsilon}$ such that $\theta_{\pm\varepsilon} \to \overline{\theta}$ as $\varepsilon \to 0$ and $I(\theta\overline{q}) > c - \varepsilon$ if and only if $\theta \in (\theta_-, \theta_+)$. In particular for each $r > 0$ there is an $\varepsilon = \varepsilon(r)$ such that $\theta \in (\theta_-, \theta_+)$ implies that $g(\theta) = \theta\overline{q} \in B_r(q)$.

At this point assume further that V satisfies one further condition

(**) There is an $\alpha > 0$ such that $\mathcal{K}^{c+\alpha}$ consists of isolated points.

Observe that the above proposition corresponds to Proposition 2.22 [**3**].Therefore we can apply the argument in [**3**] to prove the existence of infinitely many homoclinic solutions of (HS). Therefore the following theorem was essentially proved in [**3**].

THEOREM 2.2. *If V satisfies* (V_1)-(V_4) *and* (**) *,then the problem* (HS) *has infinitely many homoclinic solutions.*

Now it is natural to ask what kind of potential guarantees the condition (**), that is, the discreteness of critical points of the corresponding functional I.

We now give a condition on V whose corresponding functional satisfies (**).

THEOREM 2.3. *If V satisfies the conditions* (V_1)-(V_4), *and*

$$(V_5) \qquad \langle V_{qq}(t,q)p,p\rangle \geq \kappa|p|^2, \quad p,q \in \mathbf{R}^n, \quad \kappa > -\frac{1}{2}$$

then the critical points of I are all isolated. Therefore the problem (HS) *has infinitely many homoclinic solutions.*

Proof. Let q be a critical point of I. Thus for any $p \in E$ we have

$$\begin{aligned} 0 = \langle I'(q),p\rangle = &\int_{-\infty}^{\infty} (\langle \dot q, \dot p\rangle + \langle p, L(t)q\rangle)dt \\ &- \int_{-\infty}^{\infty} \langle p, V_q(t,q)\rangle dt. \end{aligned}$$

Now

$$\begin{aligned} I(q+p) &= \frac{1}{2}\int_{-\infty}^{\infty} (|\dot q + \dot p|^2 + \langle q+p, L(t)(q+p)\rangle)dt - \int_{-\infty}^{\infty} V(t,q+p)dt \\ &= \frac{1}{2}\int_{-\infty}^{\infty} (|\dot q|^2 + \langle q, L(t)q\rangle)dt + \int_{-\infty}^{\infty} (\langle \dot q, \dot p\rangle + \langle p, L(t)q\rangle)dt \\ &\quad + \frac{1}{2}\int_{-\infty}^{\infty} (|\dot p|^2 + \langle p, L(t)p\rangle)dt - \int_{-\infty}^{\infty} V(t,q+p)dt \\ &= I(q) + \int_{-\infty}^{\infty} V(t,q)dt + \int_{-\infty}^{\infty} \langle p, V_q(t,q)\rangle dt \\ &\quad + \frac{1}{2}\|p\|^2 - \int_{-\infty}^{\infty} V(t,q+p)dt \\ &= I(q) + \frac{1}{2}\|p\|^2 + \int_{-\infty}^{\infty} (V(t,q) + \langle p, V_q(t,q)\rangle - V(t,q+p))dt. \end{aligned}$$

Now

$$\begin{aligned}
&V(t,q)+\langle p,V_q(t,q)\rangle-V(t,q+p)\\
=&\int_0^1 s\langle p,V_{qq}(t,q+sp)p\rangle dt\\
=&\int_0^1 s\langle p,(V_{qq}(t,q+sp)-V_{qq}(t,q))p\rangle dt+\langle p,V_{qq}(t,q)p\rangle.
\end{aligned}$$

Note that $\|p\|_{L^\infty}\le\sqrt{2}\|p\|$. Hence

$$\begin{aligned}
\int_{-\infty}^{\infty}&(V(t,q)+\langle p,V_q(t,q)\rangle-V(t,q+p))dt\\
&=o(\|p\|^2)+\int_{-\infty}^{\infty}\langle p,V_{qq}(t,q)p\rangle dt.
\end{aligned}$$

Observe that

$$\begin{aligned}
\int_{-\infty}^{\infty}\langle p,V_{qq}(t,q)p\rangle dt&=\|p\|^2\int_{-\infty}^{\infty}\langle\frac{p}{\|p\|},V_{qq}(t,q)\frac{p}{\|p\|}\rangle dt\\
&=h(p)\|p\|^2.
\end{aligned}$$

We see here that h is homogeneous of degree 0 and that $h\ge\kappa>-\frac{1}{2}$ by (V_5). Hence we now have the following estimate;

$$I(q+p)=I(q)+(\frac{1}{2}+h(p))\|p\|^2+o(\|p\|^2).$$

This completes the proof.

Combining theorem 2.3 and theorem 2.3, we have

THEOREM 2.4. *If V satisfies* (V_1)–(V_5), *then the problem* (HS) *has infinitely many solutions.*

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