Comm. Korean Math. Soc. 9 (1994), No. 1, pp. 89–102

ON THE GROWTH OF MEROMORPHIC FUNCTIONS

KI-HO KWON AND KYU BUM HWANG

1. Introduction

Let $f(z)$ be meromorphic in the complex plane and denote by $n(r,f)$ the number of poles of f in $|z| \le r$. Then the Nevanlinna characteristic is defined as $T(r,f) = m(r,f) + N(r,f)$, where $m(r,f)$ is the L_1 norm of $\log^+ |f(re^{i\theta})|$ and

$$N(r,f) = \int_0^r \frac{n(t,f) - n(0,f)}{t}\, dt + n(0,f) \log r$$

(for this and other standard terminology, see [3]).

In this paper we compare the growth of $T(r,f)$ with that of

$$m_2(r,f) = \left\{ \frac{1}{2\pi} \int_0^{2\pi} (\log |f(re^{i\theta})|)^2 d\theta \right\}^{\frac{1}{2}}.$$

An upper bound for $m_2(r,f)$ in terms of the Nevanlinna characteristic was obtained by J. Miles and D. Shea in

THEOREM A [6]. *Let f be meromorphic in $|z| \le R$, with $f(0) = 1$. Then*

$$m_2(r,f) \le \left\{1 + A/\sqrt{\log(R/r)}\right\} T(R,f), \tag{1}$$

where $0 < r < R$ and $A = 8\sqrt{\log 2}$.

The following two theorems improve Theorem A when $T(R,f)/T(r,f)$ is big.

Received June 25, 1993. Revised October 8, 1993.

The first author was supported by the Wharangdae Research Institute, Korea Military Academy, 1992.

THEOREM 1. *Let $f(z)$ be meromorphic in $|z| \le \alpha r$ $(1 < \alpha \le 2,\ r > 0)$, with $f(0) = 1$. Then, for $0 < \varepsilon \le 1/2$, we have*

$$m_2(r,f) \le B(\alpha,\varepsilon) T(r,f)^{\frac{1}{2}-\varepsilon} T(\alpha r, f)^{\frac{1}{2}+\varepsilon}, \tag{2}$$

where

$$B(\alpha,\varepsilon) = \frac{6\sqrt{5}}{\sqrt{\varepsilon}(\alpha-1)^{\frac{1}{2}+\varepsilon}}.$$

We do not know if the exponent $\frac{1}{2}+\varepsilon$ on $T(\alpha r, f)$ in the inequality (2) is precise. In [4, Theorem 3.2], we showed that if in (2) we consider pairs of exponents on $T(r,f)$ and $T(\alpha r, f)$ with sum 1, then the exponent on $T(\alpha r, f)$ must be at least $1/4$.

In addition, we can get a similar result as Theorem 1 which does not contain the term ε.

THEOREM 2. *Under the same assumptions as in Theorem 1, we have*

$$m_2(r,f) \le \frac{20}{\alpha - 1}[T(r,f)T(\alpha r, f)\{2 + \log T(\alpha r, f)\}]^{1/2} \tag{3}$$

Let $S(r)$ be a real nonnegative function, then the order of the function $S(r)$ is defined as

$$\overline{\lim_{r\to\infty}} \frac{\log S(r)}{\log r}.$$

The order of a meromorphic function f is defined as the order of $T(r,f)$.

We now seek an upper bound for $m_2(r,f)/T(r,f)$ as a function of r.

COROLLARY. *Let $f(z)$ be meromorphic in the plane of order λ, $0 < \lambda < \infty$. Then for any $\varepsilon > 0$ there exists a positive real number $r(\varepsilon)$ such that*

$$m_2(r,f) \le r^{\frac{\lambda}{2}+\varepsilon} T(r,f), \quad r > r(\varepsilon). \tag{4}$$

The inequality (4) shows that $m_2(r,f)/T(r,f)$ has order at most $\lambda/2$. We do not know if the inequality (4) is sharp. But the exponent on r in (4) must be greater than or equal to $\lambda/4$ by [4, Theorem 3.2].

The corollary follows immediately from Theorem 2. In fact, without loss of generality we may assume that $|f(0)| = 1$. Then we have, by

(3) with $\alpha = 2$ and the fact that $T(R, f)$ is an increasing function of R [3, p8],

$$\frac{m_2(r,f)}{T(r,f)} \le 20[T(2r,f)\{2+\log T(2r,f)\}]^{1/2}.$$

Since

$$T(2r,f) = O(r^{\lambda+\varepsilon}), \quad r \to \infty,$$

we have

$$\frac{m_2(r,f)}{T(r,f)} \le O(r^{\frac{\lambda}{2}+\varepsilon}), \quad r \to \infty.$$

We next consider the opposite direction of the inequality (1). If f is a meromorphic function in the plane, then it is in general not true for any fixed constants A and B that for all $r > 0$,

$$T(r,f) \le Am_2(Br,f)$$

(consider the function $f(z) = (z+1)/(z-1)$).

In case f is entire, it is easy to see that

$$T(r,f) \le m_2(r,f). \tag{5}$$

Hence $T(r,f)$ and $m_2(r,f)$ have the same order for entire f by (1) and (5).

Let $f(z)$ be meromorphic of finite order λ, and let $q = [\lambda]$. Assume for convenience that $f(0) = 1$ and define $\{\alpha_m\}$ by

$$\log f(z) = \sum_{m=1}^{\infty} \alpha_m z^m$$

for z near 0. Write

$$f(z) = e^{p(z)} \prod E\left(\frac{z}{z_\nu}, q\right) \Big/ \prod E\left(\frac{z}{w_\nu}, q\right),$$

where $z_\nu \ne w_\nu$, $p(z) = \alpha_q z^q + \cdots + \alpha_1 z$ and

$$E(x,q) = (1-x)\exp(x + x^2/2 + \cdots + x^q/q).$$

Let $c_m(r,f)$ be the m-th Fourier coefficient of $\log|f(re^{i\theta})|$:

$$c_m(r,f) = \frac{1}{2\pi}\int_0^{2\pi} (\log|f(re^{i\theta})|)e^{-im\theta}d\theta.$$

Then

$$m_2(r,f)^2 = \sum_{m=-\infty}^{\infty} |c_m(r,f)|^2.$$

Edrei and Fuchs [2] had shown that

$$c_m(r,f) = \frac{1}{2}\alpha_m r^m + \frac{1}{2m}\sum_{|z_\nu|\le r}\left\{\left(\frac{r}{z_\nu}\right)^m - \left(\frac{\overline{z_\nu}}{r}\right)^m\right\} - \frac{1}{2m}\sum_{|w_\nu|\le r}\left\{\left(\frac{r}{w_\nu}\right)^m - \left(\frac{\overline{w_\nu}}{r}\right)^m\right\} \tag{6}$$

for $m \ge 1$ and, for $m \ge q+1$; also

$$c_m(r,f) = -\frac{1}{2m}\left\{\sum_{|z_\nu|>r}\left(\frac{r}{z_\nu}\right)^m - \sum_{|w_\nu|>r}\left(\frac{r}{w_\nu}\right)^m + \sum_{|z_\nu|\le r}\left(\frac{\overline{z_\nu}}{r}\right)^m - \sum_{|w_\nu|\le r}\left(\frac{\overline{w_\nu}}{r}\right)^m\right\}.$$

Obviously, $c_m(r,f) = \overline{c_{-m}(r,f)}$ for $m \le -1$ and

$$c_0(r,f) = N(r,1/f) - N(r,f) \le T(r,f). \tag{7}$$

In particular, if $p(z) = 0$ and $|\arg z_\nu| \le \omega$, $|\pi - \arg w_\nu| \le \omega$ with $0 \le \omega \le (\pi - \varepsilon)/2q$, $\varepsilon > 0$, then M. Ozawa indicated in [7] for meromorphic functions of positive genus that

$$T(r,f) \le C(q,\omega)m_2(r,f)$$

for some constant $C(q,\omega)$ depending only on q and ω. We can generalize Ozawa's result in

THEOREM 3. *Let* $q \geq 1$ *and let*

$$f(z) = \prod E\left(\frac{z}{z_\nu}, q\right) \Big/ \prod E\left(\frac{z}{w_\nu}, q\right),$$

where $|\arg z_\nu| \leq \omega$, $|\pi - \arg w_\nu| \leq \omega$ *with* $0 \leq \omega \leq \frac{\pi}{2} - \varepsilon$, $\varepsilon > 0$. *Then we have for* $r > 0$,

$$T(r, f) \leq \frac{1 + \cos\omega}{2\cos\omega} m_2(r, f).$$

REMARKS.

1) We deduce from Theorem A and Theorem 3 that for any meromorphic function satisfying the hypotheses in theorem 3, $T(r, f)$ and $m_2(r, f)$ have the same order.
2) Theorem 3 shows that if $\omega = 0$, i.e., f has its zeros and poles in the positive and negative real axis respectively, $T(r, f) \leq m_2(r, f)$.

2. Proof of Theorem 1

For $m \geq 1$ and $\beta > 1$, (6) gives

$$\begin{aligned} &c_m(r, f) - \beta^{-m} c_m(\beta r, f) \\ &= -\frac{1}{2m} \sum_{r < |z_\nu| \leq \beta r} \left\{ \left(\frac{r}{z_\nu}\right)^m - \left(\frac{\overline{z_\nu}}{\beta^2 r}\right)^m \right\} - \frac{1}{2m} \sum_{|z_\nu| \leq r} \left(\frac{\overline{z_\nu}}{r}\right)^m (1 - \beta^{-2m}) \\ &\quad + \frac{1}{2m} \sum_{r < |w_\nu| \leq \beta r} \left\{ \left(\frac{r}{w_\nu}\right)^m - \left(\frac{\overline{w_\nu}}{\beta^2 r}\right)^m \right\} + \frac{1}{2m} \sum_{|w_\nu| \leq r} \left(\frac{\overline{w_\nu}}{r}\right)^m (1 - \beta^{-2m}). \end{aligned}$$

Hence we obtain, for $m \geq 1$ and $\beta > 1$,

$$|c_m(r, f)| \leq \beta^{-m} |c_m(\beta r, f)| + \frac{1}{2m}(1 - \beta^{-2m}) n(\beta r),$$

where $n(R) = n(R, 1/f) + n(R, f)$. Using $n(R)\log\beta \leq N(\beta R)$ and

$$|c_m(R, f)| \leq \frac{1}{2\pi} \int_0^{2\pi} \left|\log|f(Re^{i\theta})|\right| d\theta \leq 2T(R, f), \tag{8}$$

we get, for $m \geq 1$ and $\beta > 1$,

$$|c_m(r,f)| \leq 2T(\beta^2 r, f)\left(\beta^{-m} + \frac{1}{2m\log\beta}\right). \tag{9}$$

For any $0 < \varepsilon \leq 1/2$, (7) gives

$$|c_0(r,f)|^2 \leq T(r,f)^2 \leq T(r,f)^{1-2\varepsilon}T(\beta^2 r,f)^{1+2\varepsilon}$$

and, for $m \geq 1$,

$$\begin{aligned}|c_m(r,f)|^2 &\leq |c_m(r,f)|^{1-2\varepsilon}\left\{2T(\beta^2 r,f)(\beta^{-m} + \frac{1}{2m\log\beta})\right\}^{1+2\varepsilon}\\ &\leq 4T(r,f)^{1-2\varepsilon}T(\beta^2 r,f)^{1+2\varepsilon}\left(\beta^{-m} + \frac{1}{2m\log\beta}\right)^{1+2\varepsilon}.\end{aligned}$$

Thus we have

$$\begin{aligned}m_2(r,f)^2 &= |c_0(r,f)|^2 + 2\sum_{m=1}^{\infty}|c_m(r,f)|^2\\ &\leq \{1 + 8D(\beta,\varepsilon)\}T(r,f)^{1-2\varepsilon}T(\beta^2 r,f)^{1+2\varepsilon},\end{aligned} \tag{10}$$

where

$$D(\beta,\varepsilon) = \sum_{m=1}^{\infty}\left(\beta^{-m} + \frac{1}{2m\log\beta}\right)^{1+2\varepsilon}.$$

We observe that, by the Minkowski inequality,

(11)

$$\begin{aligned}D(\beta,\varepsilon)^{\frac{1}{1+2\varepsilon}} &\leq \left\{\sum_{m=1}^{\infty}\beta^{-m(1+2\varepsilon)}\right\}^{\frac{1}{1+2\varepsilon}} + \left\{\sum_{m=1}^{\infty}\frac{1}{(2m\log\beta)^{1+2\varepsilon}}\right\}^{\frac{1}{1+2\varepsilon}}\\ &\leq \left(\frac{1}{\beta^{1+2\varepsilon}-1}\right)^{\frac{1}{1+2\varepsilon}} + \frac{1}{2\log\beta}\left(\frac{1+2\varepsilon}{2\varepsilon}\right)^{\frac{1}{1+2\varepsilon}}.\end{aligned}$$

Since $a^p + b^p \le (a+b)^p \le 2p(a^p + b^p)$ for $a > 0$, $b > 0$, and $p > 1$, and since $1 < \beta \le \sqrt{2}$ implies $\log \beta^2 \ge (\beta^2 - 1)/2$, we obtain from (11) for $0 < \varepsilon \le 1/2$ and $1 < \beta \le \sqrt{2}$ that

(12)

$$\begin{aligned} D(\beta,\varepsilon) &\le 2^{1+2\varepsilon} \left\{ \frac{1}{\beta^{1+2\varepsilon} - 1} + \frac{1+2\varepsilon}{2\varepsilon} \left(\frac{1}{2\log\beta}\right)^{1+2\varepsilon} \right\} \\ &\le 4 \left\{ \frac{\beta^{1+2\varepsilon} + 1}{\beta^{2(1+2\varepsilon)} - 1} + \frac{1}{\varepsilon} \left(\frac{2}{\beta^2 - 1}\right)^{1+2\varepsilon} \right\} \\ &\le 4 \left\{ \frac{3}{(\beta^2 - 1)^{1+2\varepsilon}} + \frac{4}{\varepsilon(\beta^2 - 1)^{1+2\varepsilon}} \right\} = \frac{4(4+3\varepsilon)}{\varepsilon(\beta^2 - 1)^{1+2\varepsilon}}. \end{aligned}$$

Taking $\alpha = \beta^2$, we deduce from (10) and (12) that for $1 < \alpha \le 2$, $0 < \varepsilon \le 1/2$ and $r > 0$,

$$m_2(r,f) \le B(\alpha,\varepsilon) T(r,f)^{\frac{1}{2}-\varepsilon} T(\alpha r, f)^{\frac{1}{2}+\varepsilon},$$

where

$$B(\alpha,\varepsilon) = \left\{ 1 + \frac{32(4+3\varepsilon)}{\varepsilon(\alpha - 1)^{1+2\varepsilon}} \right\}^{\frac{1}{2}} \le \frac{6\sqrt{5}}{\sqrt{\varepsilon}(\alpha - 1)^{1/2+\varepsilon}}.$$

3. Proof of Theorem 2

It follows from (9) that

(13)

$$\begin{aligned} \sum_{m=1}^{\infty} |c_m(r,f)|^2 &\le 2 \sum_{m=1}^{\infty} |c_m(r,f)| T(\beta^2 r, f) \left(\beta^{-m} + \frac{1}{2m\log\beta} \right) \\ &= A + B + C, \end{aligned}$$

where

$$A = 2 \sum_{m=1}^{\infty} |c_m(r,f)| T(\beta^2 r, f) \beta^{-m},$$

$$B = \sum_{m=1}^{N} |c_m(r,f)| T(\beta^2 r, f) \frac{1}{m \log \beta}$$

and

$$C = \sum_{m=N+1}^{\infty} |c_m(r,f)| T(\beta^2 r, f) \frac{1}{m \log \beta}.$$

Since $1 < \beta \leq \sqrt{2}$ implies $\log \beta \geq (\beta - 1)/2$, we then have, by (8) and (9),

(14)

$$\begin{aligned} A &\leq 4T(r,f)T(\beta^2 r, f) \frac{\beta^{-1}}{1-\beta^{-1}} \\ &= \frac{4}{\beta - 1} T(r,f) T(\beta^2 r, f), \\ B &\leq \frac{2}{\log \beta} T(r,f) T(\beta^2 r, f) \sum_{m=1}^{N} \frac{1}{m} \\ &\leq \frac{4}{\beta - 1} T(r,f) T(\beta^2 r, f)(1 + \log N) \end{aligned}$$

and

$$\begin{aligned} C &\leq \frac{2}{\log \beta} T(\beta^2 r, f)^2 \sum_{m=N+1}^{\infty} \frac{1}{m} \left(\beta^{-m} + \frac{1}{2m \log \beta} \right) \\ &\leq \frac{4}{\beta - 1} T(\beta^2 r, f)^2 \left\{ \frac{1}{N+1} \cdot \frac{\beta^{-N-1}}{1 - \beta^{-1}} + \frac{1}{\beta - 1} \sum_{m=N+1}^{\infty} \frac{1}{m^2} \right\} \\ &\leq \frac{4}{(\beta - 1)^2} T(\beta^2 r, f)^2 \left\{ \frac{1}{N+1} + \frac{1}{N} \right\}. \end{aligned}$$

Now choosing $N = [T(\beta^2 r, f)] + 1$, we have

$$\text{(15)} \qquad \begin{aligned} B &\leq \frac{4}{\beta - 1} T(r,f) T(\beta^2 r, f) \{2 + \log T(\beta^2 r, f)\}, \\ C &\leq \frac{8}{(\beta - 1)^2} T(\beta^2 r, f). \end{aligned}$$

Hence we deduce from (7), (13), (14) and (15) that

$$\begin{aligned} m_2(r,f)^2 &= |c_0(r,f)|^2 + 2\sum_{m=1}^{\infty} |c_m(r,f)|^2 \\ &\le T(r,f)^2 + 2\frac{16}{(\beta-1)^2} T(r,f)T(\beta^2 r,f)\{2+\log T(\beta^2 r,f)\} \\ &\le \frac{250}{(\beta^2-1)^2} T(r,f)T(\beta^2 r,f)\{2+\log T(\beta^2 r,f)\}. \end{aligned}$$

Taking $\alpha = \beta^2$, we conclude that

$$m_2(r,f) \le \frac{20}{\alpha-1}[T(r,f)T(\alpha r,f)\{2+\log T(\alpha r,f)\}]^{\frac{1}{2}}.$$

4. Proof of Theorem 3

Let $q \ge 1$ and let

$$f(z) = \prod E\left(\frac{z}{z_\nu}, q\right) \Big/ \prod E\left(\frac{z}{w_\nu}, q\right),$$

where $|\arg z_\nu| \le \omega$, $|\pi - \arg w_\nu| \le \omega$ with $0 \le \omega \le \frac{\pi}{2} - \varepsilon$, $\varepsilon > 0$. Then we have, by (6),

$$c_1(r,f) = \frac{1}{2}\sum_{|z_\nu|\le r}\left\{\frac{r}{z_\nu} - \frac{\overline{z_\nu}}{r}\right\} - \frac{1}{2}\sum_{|w_\nu|\le r}\left\{\frac{r}{w_\nu} - \frac{\overline{w_\nu}}{r}\right\}.$$

Hence

$$\begin{aligned} \text{(16)} \qquad \operatorname{Re} c_1(r,f) &\ge \frac{1}{2}\sum_{|z_\nu|\le r}\left\{\frac{r}{|z_\nu|} - \frac{|z_\nu|}{r} + \frac{r}{|w_\nu|} - \frac{|w_\nu|}{r}\right\}\cos\omega \\ &= \frac{\cos\omega}{2}\int_0^r \left(\frac{r}{t} - \frac{t}{r}\right) dn(t), \end{aligned}$$

where $n(t) = n(t,f) + n(t,1/f)$. Integration by parts applied twice to (16) yields

$$\begin{aligned} \frac{\operatorname{Re} c_1(r,f)}{\cos\omega} &\ge N(r,f) + N(r,\frac{1}{f}) + \frac{1}{2}\int_0^r \frac{N(r,f)+N(r,1/f)}{t}\left\{\frac{r}{t} - \frac{t}{r}\right\} dt \\ &\ge N(r,f) + N(r,\frac{1}{f}). \end{aligned}$$

Thus we get

$$(17)\quad m_2(r,f) \geq |c_1(r,f)| \geq \operatorname{Re} c_1(r,f) \geq \left\{N(r,f) + N(r,\frac{1}{f})\right\}\cos\omega.$$

It is clear that

$$(18)\qquad m_2(r,f) \geq m(r,f) + m(r,\frac{1}{f}).$$

Therefore we deduce from (17) and (18) that

$$\begin{aligned}(1+\cos\omega)m_2(r,f) &\geq \left\{T(r,f) + T(r,\frac{1}{f})\right\}\cos\omega \\ &= 2\cos\omega T(r,f).\end{aligned}$$

Hence we conclude that

$$T(r,f) \leq \frac{1+\cos\omega}{2\cos\omega}\, m_2(r,f).$$

5. An upper bound for $m_2(r,f)/T(r,f)$ on a set of r with positive lower logarithmic density

In the remaining part of the present paper, we seek upper estimates for $m_2(r,f)$ in terms of $T(r,f)$, now however permitting exceptional sets of r.

For $E \subset [1.\infty)$, define the logarithmic measure of E by

$$m_\ell(E) = \int_E \frac{dt}{t}.$$

The upper and lower logarithmic density of E are defined by

$$\overline{\log dens}\, E = \limsup_{r\to\infty} \frac{m_\ell(E\cap[1,r])}{\log r},$$

$$\underline{\log dens}\, E = \liminf_{r\to\infty} \frac{m_\ell(E\cap[1,r])}{\log r}.$$

We denote the Ahlfors-Shimizu characteristic by

$$T_0(r,f) = \int_0^r \frac{A(t,f)}{t}\, dt,$$

where $A(t,f)$ is the average number of solutions of $f(z) = a$ in $|z| \leq t$ as a varies over the Riemann sphere.

In 1969, Petrenko [8] proved Paley's conjecture:

THEOREM B. *For any meromorphic function f of order $\lambda < \infty$, we have*

$$\liminf_{r\to\infty}\frac{\log M(r,f)}{T(r,f)} \le \begin{cases} \frac{\pi\lambda}{\sin \pi\lambda}, & \lambda \le \frac{1}{2} \\ \pi\lambda, & \lambda > \frac{1}{2} \end{cases}$$

We now obtain a theorem for $m_2(r,f)$ analogous to Theorem B.

THEOREM 4. *Let $f(z)$ be meromorphic in the plane of order λ, $0 < \lambda < \infty$. Then there exists a set $E \subset [1,\infty)$ with positive lower logarithmic density such that*

$$\limsup_{r\to\infty, r\in E}\frac{m_2(r,f)}{T(r,f)} \le c_\lambda,$$

where c_λ is a constant depending only on λ and

$$c_\lambda = O(\sqrt{\lambda}), \quad \text{as} \quad \lambda \to \infty.$$

Proof. To prove the theorem, we need the following results from [5].

LEMMA C. *Let $f(z)$ be meromorphic in the plane of order λ, $0 < \lambda < \infty$. For $K > 1$, let*

$$E_1(K) = \{r > 1 : A(r,f)/T_0(r,f) > K\lambda\}.$$

Then we have

(a) $$\overline{\log dens}\, E_1(K) \le 1/K$$

and

(b) *if $\varepsilon > 0$, there exists $c(\varepsilon) > 0$ and a set $E_2(\varepsilon) \subset [1,\infty)$ with*

$$\underline{\log dens}\, E_2(\varepsilon) \ge c(\varepsilon)$$

such that for all $r \in E_2(\varepsilon)$,

$$T_0(re^h, f) < h(e+\varepsilon)A(r,f),$$

where $h = T_0(r,f)/A(r,f)$.

We may first assume $f(0) = 1$ for convenience. By Theorem A with $R = re^h$, we have

$$m_2(r,f) \le \{1 + 8\sqrt{\log 2}\sqrt{A(r,f)/T_0(r,f)}\}T(re^h,f). \tag{19}$$

By Lemma C (b), there exist a number $c(e) > 0$ and a set $E_2(e) \subset [1,\infty)$ with

$$\underline{\log dens}\, E_2(e) \ge c(e) > 0 \tag{20}$$

such that if $r \in E_2(e)$,

$$T_0(re^h,f) < 2eT_0(r,f). \tag{21}$$

If we choose a number K_0 so large that

$$1/K_0 < c(e), \tag{22}$$

then by Lemma C (a) there exists a set $E_1(K_0)$ with

$$\overline{\log dens}\, E_1(K_0) \le 1/K_0 \tag{23}$$

such that for $r \in [1,\infty) - E_1(K_0)$,

$$\frac{A(r,f)}{T_0(r,f)} \le K_0\lambda. \tag{24}$$

Setting $E = E_2(e) - E_1(K_0)$, we then have by (20), (22), and (23) that

$$\underline{\log dens}\, E \ge c(e) - 1/K_0 > 0,$$

and we conclude from (19), (21), and (24) that for sufficiently large $r \in E$,

$$m_2(r,f) \le \big(1 + 8\sqrt{\log 2}\sqrt{K_0\lambda}\big)(2eT(r,f)),$$

since $T(R,f) = T_0(R,f) + O(1)$ as $R \to \infty$. Hence

$$\limsup_{r\to\infty, r\in E} \frac{m_2(r,f)}{T(r,f)} \le 2e\big(1 + 8\sqrt{\log 2}\sqrt{K_0\lambda}\big) = O(\sqrt{\lambda}), \quad \lambda \to \infty.$$

This proves Theorem 4.

6. Examples

In Theorem 4 our estimate for c_λ is certainly not best possible, but at least we need

$$c_\lambda \geq \frac{\pi\sqrt{\lambda}}{2},$$

as is shown by the following example. For $0 < \alpha < 1$, we set

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(1+\alpha n)}.$$

Then $E_\alpha(z)$, called Mittag-Leffler's function, is an entire function of order $1/\alpha$ and has the following property [1, p50]: If

$$0 < \alpha < 1, \; z = re^{i\theta}, \quad r \geq 2,$$

we have that, as $r \to \infty$,

$$E_\alpha(z) = \begin{cases} \frac{1}{\alpha}\exp(z^{1/\alpha}) + o(1), & |\theta| \leq \frac{3}{4}\alpha\pi \\ o(1), & \text{otherwise.} \end{cases}$$

Hence we get

$$\begin{aligned} (25) \qquad T(r, E_\alpha) &= \frac{1}{2\pi}\int_{-\frac{3}{4}\alpha\pi}^{\frac{3}{4}\alpha\pi} \log^+ \left|\exp\{r^{\frac{1}{\alpha}} e^{i(\frac{\theta}{\alpha})}\}\right| d\theta + O(1) \\ &= \frac{1}{2\pi}\int_{-\frac{\alpha\pi}{2}}^{\frac{\alpha\pi}{2}} r^{\frac{1}{\alpha}} \cos\frac{\theta}{\alpha} d\theta + O(1) \\ &= \frac{\alpha}{\pi} r^{\frac{1}{\alpha}} + O(1), \quad r \to \infty, \end{aligned}$$

and if $m_2^+(r, f)$ is the L_2 norm of $\log^+|f(re^{i\theta})|$ then

$$\begin{aligned} (26) \quad m_2^+(r, E_\alpha)^2 &\geq (1 - o(1))\frac{1}{2\pi}\int_{-\frac{\alpha\pi}{2}}^{\frac{\alpha\pi}{2}} \left[\log^+|\exp\{r^{\frac{1}{\alpha}} e^{i(\frac{\theta}{\alpha})}\}|\right]^2 d\theta \\ &= (1 - o(1))\frac{1}{2\pi}\int_{-\frac{\alpha\pi}{2}}^{\frac{\alpha\pi}{2}} (r^{\frac{1}{\alpha}} \cos\frac{\theta}{\alpha})^2 d\theta \\ &= (1 - o(1))\frac{\alpha}{4} r^{\frac{2}{\alpha}}, \quad r \to \infty. \end{aligned}$$

Setting $\lambda = 1/\alpha$, we conclude from (25) and (26) for the entire function $E_\alpha(z)$ of order λ that

$$\liminf_{r\to\infty} \frac{m_2^+(r, E_\alpha)}{T(r, E_\alpha)} \geq \frac{\pi\sqrt{\lambda}}{2}.$$

This proves our assertion since $m_2(r, f) = m_2^+(r, f) + m_2^+(r, 1/f)$.

References

1. M. L. Cartwright, *Integral functions*, Cambridge University Press, New York, 1956.
2. A. Edrei and W. H. J. Fuchs, *On the growth of meromorphic functions with several deficient values*, Trans. Amer. Math. Soc. **93** (1959), 292–328.
3. W. K. Hayman, *Meromorphic functions*, Oxford University Press, Oxford, 1964.
4. K. Kwon, *On the growth of entire functions*, Israel J. Math., (to appear).
5. S. Hellerstein, J. Miles and J. Rossi, *On the growth of solutions of $f'' + gf' + hf = 0$*, Trans. Amer. Math. Soc. **324** (1991), 693–706.
6. J. Miles and D. F. Shea, *On the growth of meromorphic functions having at least one deficient value*, Duke Math. J. **43** (1976), 171–185.
7. M. Ozawa, *On the growth of meromorphic functions*, Kodai Math. J. **6** (1983), 250–260.
8. V. P. Petrenko, *The growth of meromorphic functions of finite lower order*, Izv. Ak. Nauk U.S.S.R. **33** (1969), 414–454.

Department of Mathematics
Korea Military Academy
P O Box 77, Gongneung, Nowon
Seoul 139–799, Korea