Comm. Korean Math. Soc. 9 (1994), No. 1, pp. 67–77

EMBEDDING THEOREM ON LIPSCHITZ SPACES

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1. Introduction

This paper is the second of a series in which smoothness properties of functions in several variables are discussed [2]. The germ of this development is in a group of papers by Taibleson [1] in which properties of functions defined on the Euclidean space are analyzed. His ideas are extended here to problems which deal with functions defined on Heisenberg group, H^n. The groundwork for such an extension has already been laid in a paper of Folland [3]. In so far as we wish to show the connection between various function spaces, it will not be amiss to discuss some of these notions. We have first of all the Lipschitz spaces $\mathrm{Lip}(\alpha, p)$. We say that $f(x) \in L^p(R^n)$ is in $\mathrm{Lip}(\alpha, p)$, $0 < \alpha \leq 1$ if

$$\sup_{h \in R^n} |h|^{-\alpha} \|f(x+h) - f(x)\|_p < \infty.$$

Closely related are the spaces of "potentials" or "fractional integrals". We have the space of Bessel potentials on the Heisenberg group introduced by Folland [3].

2. The Heisenberg group

DEFINITION. The Heisenberg group H^n is the lie group of real dimension $2n+1$, whose underlying space is $R \times C^n$, and whose group law is given by

$$(t, z)(t', z') = (t + t' + 2\,\mathrm{Im}\, z\bar{z}', z + z').$$

Its Lie algebra is generated by the left invariant vector fields X_j, Y_j, T, $j = 1, \dots, n$, given by

$$X_j = \frac{\partial}{\partial x_j} + 2y_j \frac{\partial}{\partial t},\ Y_j = \frac{\partial}{\partial y_j} - 2x_j \frac{\partial}{\partial t},\ T = \frac{\partial}{\partial t}.$$

Received June 5, 1993. Revised August 27, 1993.
This work was supported by the Korean Ministry of Education, 1991.

It is easy to verify the following commutation relations:

$$[X_j, Y_k] = 4\delta_{j,k}T,$$

and all other brackets are zero. On H^n there are a family of dilations $\gamma_r(t,z)$ that gives rise to a one parameter group of automorphisms on H^n given by $\gamma_r(t,z) = (r^2t, rz)$.

The homogeneous dimension of H^n is $Q = 2n+2$. (Folland [3]) We define the norm on H^n by

$$|(t,z)| = (t^2 + |z|^4)^{1/4}.$$

3. The sublaplacian

Let

$$\mathcal{L} = -\sum_{j=1}^{n}(X_j^2 + Y_j^2).$$

The operator $\mathcal{L}$ is homogeneous of degree 2, and $\mathcal{L}^t = \mathcal{L}$.
In the Euclidean space a fundamental solution to Δ is given by

$$E = \frac{\Gamma(n/2)}{2\pi^{n/2}(n-2)}|x|^{2-n}, \quad n \geq 3.$$

Analogously the fundamental solution to $\mathcal{L}$ is given by

$$\phi = C(|z|^4 + t^2)^{-n/2}, \quad C = \frac{2^{2-2n}\pi^{n+1}}{\Gamma(n/2)^2}.$$

(See Folland [3]). Thus $\mathcal{L}$ is locally solvable. The convolution of two functions f and g in H^n is defined by

$$f * g(u) = \int_{H^n} f(v)g(v^{-1}u)dv.$$

4. The Bessel potential and the spaces S^p_α

The definition of the Bessel potential is from Folland [3]. The principal tool is the diffusion semigroup H_t generated by -$\mathcal{L}$.

There is a unique semigroup $\{H_t,\ 0 < t < \infty\}$ of linear operators on $L^1 + L^\infty$ satisfying the following:

(1) $H_t f = f * h_t$, where $h_t(x) = h(x,t)$ is C^∞ away from 0, and on $H^n \times (0,\infty)$, $\int_{H^n} h_t(x)dx = 1$ for all t. Also for all t and x, $h(x,t) \geq 0$ and $h(rx, r^2t) = r^{-Q}h(x,t)$.

(2) If $u \in C_0^\infty$, then

$$\lim_{t\to 0} \|t^{-1}(H_t u - u) + \mathcal{L}u\|_\infty = 0.$$

(3) H_t is self adjoint, i.e., $H_t|_{L^p} = H_t|_{L^q}$, $\frac{1}{p} + \frac{1}{q} = 1$.

(4) $f \geq 0 \Longrightarrow H_t f \geq 0$, $H_t 1 = 1$. Now if we extend $h(x,t)$ to be 0 for $t \leq 0$, then h is the fundamental solution to $\mathcal{L} + \frac{\partial}{\partial t}$.

Now inspired from the classical case, we define the Bessel potential

$$J_\alpha(x) = \frac{1}{\Gamma(\alpha/2)} \int_0^\infty t^{\alpha/2-1} e^{-t} h(x,t)dt.$$

If $f \in L^p,\ 1 < p < \infty$, then $(I + \mathcal{L})^{-\alpha/2} f = f * J_\alpha$.

So define S^p_α to be the image of L^p under the operator $(I + \mathcal{L})^{-\alpha/2}$. We have the following properties of J_α. (See [3])

(1) J_α is defined for all $x \neq 0$ and even for $x = 0$ when $\alpha > Q$. J_α is C^∞ away from 0.

(2) As $x \to 0$,

$$\begin{aligned}|J_\alpha(x)| &= O(|x|^{\alpha-Q}) \text{ if } \alpha < Q\\ &= O(log\frac{1}{|x|}) \text{ if } \alpha = Q.\end{aligned}$$

(3) As $x \to \infty$, $\quad |J_\alpha(x)| = O(|x|^{-N})$ for all N. Hence, $J_\alpha \in L^1$ for all $\alpha > 0$.

The spaces $\Lambda^{p,q}_\alpha(H^n)$ is defined to be the space of those functions in $L^p(H^n)$ for which the following quantity is finite.

$$\int_{H^n} \frac{1}{|u|^{Q+\alpha q}} [\int_{H^n} |f(uv) - f(v)|^p dv]^{q/p} du < \infty, \quad 0 < \alpha < 1.$$

$$\int_{H^n} \frac{1}{|u|^{Q+\alpha q}}[\int_{H^n} |f(uv)+f(uv^{-1})-2f(v)|^p dv]^{q/p} du < \infty, \quad 1 \le \alpha.$$

When $q=\infty$, the above expression is interpreted in the normal limiting way, namely,

$$\|f\|_p + \sup_{|u|>0} \frac{\|f(uv)-f(v)\|_p}{|u|^\alpha} < \infty.$$

LEMMA 1. *$J_\alpha(v) \in L^p(H^n)$ when $\alpha - Q/p' > 0$ $(1/p + 1/p' = 1)$.*

Proof. Since when $|v|>1$ we have rapid decrease, we are interested only when $|v|<1$. Then

$$\int_{|v|<1} |J_\alpha(v)|^p dv \le C \int_{|v|<1} |v|^{p(\alpha-Q)} dv.$$

For this to be integrable we need

$$p(\alpha - Q) > -Q, \text{ i.e., } \alpha > \frac{Q}{p'}.$$

THEOREM 1. *$J_\alpha(u) \in \Lambda^{p,\infty}_{\alpha-\frac{Q}{p'}}(H^n)$ when $0 < \alpha - \frac{Q}{p'} < 1$, $\frac{1}{p}+\frac{1}{p'}=1$.*

Proof. For simplicity we will prove the theorem when $n=1$. Looking at the size of J_α, the case $|u|\ge 1$ is contained in the definition. So we will focus when $|u|<1$. Setting $u=(t,x,y)$, $v=(\tau,-x',z)$, we are going to estimate the L^p norm first. Since

$$\begin{aligned}|J_\alpha(uv)-J_\alpha(v)| &\le |u| \sup_{\rho\in[0,1]} |\nabla J_\alpha((\rho u)v)| \\ &= |u| \sup_{\rho} |\nabla J_\alpha(\rho^2 t + \tau + 2\rho y x', \rho x - x', \rho y + z)|.\end{aligned}$$

We will divide into two cases (A) when $|x'|<1$ and (B) when $|x'|\ge 1$ and estimate J_α separately. First we will consider the case (A) $|x'|<1$. Case (A) is divided into 3 types according to the size of u and x'.

Type (I) $|u| < \frac{1}{100}|x'|, |\tau|^{1/2} + |z| < |x'|$,
Type (II) $|u| < \frac{1}{100}|x'|, |\tau|^{1/2} + |z| > |x'|$, and
Type (III) $|u| < \frac{1}{100}|x'|$
Note that for type (I) and (II), the quantity

$$|\rho^2 t 2\rho y x' + \tau|^{1/2} + |\rho x - x'| + |\rho y + z|$$

is comparable to $|\tau|^{1/2} + |x'| + |z|$, and so

$$\begin{aligned} &\|\nabla J_\alpha((\rho u)v)\|^p_{L^p(100|u|<|x'|<1)} \\ &\leq C \int\int\int_{100|u|<|x'|<1} (|\tau|^{1/2} + |x'| + |z|)^{p(\alpha-4-1)} d\tau dz dx' \\ &= (\mathrm{I}) + (\mathrm{II}). \end{aligned}$$

$$(\mathrm{I}) = \int\int\int_{100|u|<|x'|<1, |\tau|^{1/2}+|z|<|x'|} (|\tau|^{1/2} + |x'| + |z|)^{p(\alpha-5)} d\tau dz dx'$$

and

$$(\mathrm{II}) = \int\int\int_{100|u|<|x'|<1, |\tau|^{1/2}+|z|>|x'|} (|\tau|^{1/2} + |x'| + |z|)^{p(\alpha-5)} d\tau dz dx'.$$

Estimate of (I) is simple.

$$(1) \qquad (\mathrm{I}) \leq \int_{100|u|<|x'|<1} |x'|^{p(\alpha-5)} |x'|^3 dx'.$$

On the other hand, to estimate (II), we divide the range into shells of size roughly $2^j|x'|$, $j = 0, 1, 2, \ldots$. Then

(2)

$$\begin{aligned} (\mathrm{II}) &\leq C \sum_{j=0} \int\int\int_{100|u|<|x'|<1, |\tau|^{1/2}+|z|<2^j|x'|} (2^j|x'|)^{p(\alpha-5)} d\tau dz dx' \\ &\leq C \sum_{j=0} \int_{100|u|<|x'|<1} (2^j|x'|)^{p(\alpha-5)+3} dx' \\ &= C \sum_{j=0} |2^j|^{p(\alpha-5)+3} \int_{100|u|<|x'|<1} |x'|^{p(\alpha-5)+3} dx'. \end{aligned}$$

For the integral to converge we need the condition $p(\alpha - 5) + 3 < -1$ which is equivalent to $\alpha - 4/p' < 1$. By the same token the geometric series converges.

To estimate type (III) when $|x'| < 100|u|$, $|x'| < 1$ we use triangle inequality to obtain

$$(\mathrm{III}) = \|J_\alpha(uv) - J_\alpha(v)\|^p_{L^p(|x'|<100|u|,|x'|<1)}$$
$$\leq C \int\int\int_{|x'|<100|u|,|x'|<1} (|\tau|^{1/2} + |x'| + |z|)^{p(\alpha-4)} d\tau dz dx'.$$

Using the same method as in the case $100|u| < |x'|$, we find the above integral is bounded by

$$C \int_{|x'|<100|u|} |x'|^{p(\alpha-4)+3} dx'. \tag{3}$$

From the integrability condition, we need

$$p(\alpha - 4) + 3 > -1 \quad \text{i.e., } \alpha - 4/p' > 0.$$

Finally, we have to consider the case (B) when $|x'| \geq 1$. But in that case we have rapid decrease of J_α and ∇J_α. So we have better estimate than the case $|x'| < 1$. Now that we have shown $J_\alpha \in L^p(H^n)$, we will try to show $J_\alpha \in \Lambda^{p,\infty}_{\alpha-\frac{Q}{p'}}(H^n)$, i.e., we have to show the following quantity

$$\sup_{|u|>0} \frac{\|J_\alpha(uv) - J_\alpha(v)\|_p}{|u|^{\alpha-\frac{Q}{p'}}} \quad \text{is finite.} \tag{$*$}$$

~~To estimate above we will consider only when $|x'| < 1$ because otherwise,~~ we have rapid decrease of J_α and ∇J_α and obtain better estimate than the case $|x'| < 1$. As before we will divide into two cases according to the size of $|x'|$ and $|u|$, i.e., when $|x'| > 100|u|$ and $|x'| < 100|u|$. For convenience put $\tilde{\alpha} = \alpha - \frac{Q}{p'} = \alpha - 4 + 4/p$.

Case A. When $|x'| > 100|u|$, we use equation (1) and (2). Then

$$(*) \leq C \frac{|u| \sup_\rho \|\nabla J_\alpha((\rho u)v)\|_p}{|u|^{\tilde{\alpha}}}$$
$$\leq C|u|^{1-\tilde{\alpha}}[\int_{|x'|>100|u|} |x'|^{p(\alpha-5)+3} dx']^{1/p}$$
$$\leq C|u|^{1-\tilde{\alpha}+(\alpha-5)+4/p} = CO(1).$$

Case B. When $|x'| < 100|u|$, we use equation (3). Then

$$\begin{aligned}(*) &\le C|u|^{-\tilde{\alpha}}[\int_{|x'|<100|u|} |x'|^{p(\alpha-4)+3}dx']^{1/p}\\ &\le C|u|^{-\tilde{\alpha}+(\alpha-4)+\frac{4}{p}} = C0(1).\end{aligned}$$

This proves the theorem when $n = 1$.

For the general case, we have

$$\begin{aligned}|\nabla J_\alpha(v)| &= o(|v|^{\alpha-Q-1}) \text{ as } |v| \to 0\\ &= O(|v|^{-N}) \text{ for any } N \text{ as } v \text{ goes to } \infty.\end{aligned}$$

The proof goes exactly same way as in the case $n = 1$ and we will leave it to the reader.

PROPOSITION 1. *$\mathcal{J}^\beta$ maps $\Lambda^{p,q}_\alpha(H^n)$ isomorphically onto $\Lambda^{p,q}_{\alpha+\beta}(H^n)$ when α, $\beta \ge 0$.*

Proof. Let $f \in \Lambda^{p,q}_\alpha(H^n)$. First we want to show

$$\mathcal{J}^\beta(f) = f * J_\beta \in \Lambda^{p,q}_{\alpha+\beta}(H^n)$$

To show the inclusion we need to show the following is finite.

$$(*) \qquad \int \frac{\|(J_\beta * f)(uv) - (J_\beta * f)(v)\|_p^q}{|u|^{Q+(\alpha+\beta)q}} du < \infty.$$

We are going to estimate the L^p norm inside the integral. For simplicity, we will restrict ourselves only to when $n = 1$. Let $u = (t, x, y), v = (\tau, -x', z)$. We will estimate the L^p norm separately according to the size of u, and x', i.e.

$$\|(J_\beta * f)(uv) - (J_\beta * f)(v)\|_p = (\mathrm{A}) + (\mathrm{B})$$

$$\begin{aligned}(\mathrm{A}) &= \|(J_\beta * f)(uv) - (J_\beta * f)(v)\|_{L^p(|x'|>100|u|)}\\ (\mathrm{B}) &= \|(J_\beta * f)(uv) - (J_\beta * f)(v)\|_{L^p(|x'|<100|u|)}.\end{aligned}$$

We will estimate (A) first.

(Case A) When $|x'| > 100|u|$, We can express

$$(J_\beta * f)(uv) - (J_\beta * f)(v) = u\int_0^1 \nabla(J_\beta * f)((\rho u)v)d\rho$$
$$\leq |u| \sup_{\rho\in[0,1]} |\nabla(J_\beta * f)((\rho u)v)|.$$

Then with the aid of Young's inequality,

$$(\mathrm{A}) \leq |u| \int_0^1 \|(\nabla J_\beta) * f((\rho u)v)\|_p d\rho$$
$$\leq |u| \|\nabla J_\beta((\rho u)v)\|_{L^1} \int_0^1 \|f((\rho u)v)\|_p d\rho.$$

We can compute $\|\nabla J_\beta(\rho u)v)\|_{L^1}$ following similar steps as in the proof of theorem 1 to find that

$$\|\nabla J_\beta(\rho u)v)\|_{L^1(100|u|<|x'|,dv)}$$
$$\leq C\int_{100|u|<|x'|} |x'|^{\beta-2}dx' \leq C'|u|^{\beta-2} \quad \text{for} \quad \beta < 1.$$

In this case with the aid of Young's inequality and Fubini's theorem,

$$(\mathrm{A}) \leq C|u|^\beta \|f(uv) - f(v)\|_p.$$

When $\beta \geq 1$, we use the relation $\mathcal{J}^{\alpha+\beta} = \mathcal{J}^\alpha \mathcal{J}^\beta$.

Now we will estimate (B). In this case we can apply Young's inequality directly to obtain

$$(\mathrm{B}) \leq C\|J_\beta\|_{L^1(|x'|<100|u|,dv)} \|f(uv) - f(v)\|_p.$$

But $\|J_\beta\|_{L^1(|x'|<100|u|,dv)} \leq C|u|^\beta$. So we have gained same power of u for both cases. Combining all these, we finally obtain

$$(**) \leq C\int \frac{|u|^{\beta q}\|f(uv) - f(v)\|_p^q}{|u|^{Q+(\alpha+\beta)q}} du \leq C\|f\|_{\Lambda_\alpha^{p,q}}.$$

This inequality also means $\mathcal{J}^\beta$ is also one-one. To show this map is onto, we need the following proposition from Stein [6, p153].

PROPOSITION 2. *Suppose $\alpha > 1$. Then $f \in \Lambda_\alpha^{p,q}(R^n)$ iff $f \in L^p(R^n)$ and $\frac{\partial f}{\partial x_j} \in \Lambda_{\alpha-1}^{p,q}$. The norm $\|f\|_{\Lambda_{\alpha-1}^{p,q}}$ and $\|f\|_p + \sum \|\frac{\partial f}{\partial x_j}\|_{\Lambda_{\alpha-1}^{p,q}}$ are equivalent.*

With this proposition, we can claim that the image of $\Lambda_\alpha^{p,q}$ under $\mathcal{J}^2$ is all of $\Lambda_{\alpha+2}^{p,q}$. To see this, let $f \in \Lambda_{\alpha+2}^{p,q}$. Then $f \in \Lambda_\alpha^{p,q}$; also $\mathcal{L}f \in \Lambda_\alpha^{p,q}$. Therefore $(I + \mathcal{L})f \in \Lambda_\alpha^{p,q}$. However,

$$\mathcal{J}^2[(I + \mathcal{L})f] = f.$$

This means $\mathcal{J}^2$ is onto. Because $\mathcal{J}^2$ is onto, $\mathcal{J}^{2-\beta}$ is one-one, and $\mathcal{J}^2 = \mathcal{J}^{2-\beta}\mathcal{J}^\beta$ for $0 < \beta < 2$, Then $\mathcal{J}^\beta$ must be onto for that range of β. Finally by superimposing such $\mathcal{J}^\beta$ we arrive at the conclusion that $\mathcal{J}^\beta$ is onto for any positive β and the claim is proved with the aid of the closed graph theorem.

We also know that $(I + \mathcal{L})^{-\alpha} = [(I + \mathcal{L})^\alpha]^{-1}$. (Komatsu, Theorems 7.2 and 7.3 [4], Folland, Theorem 3.15 [3]) From this we can deduce if $\alpha,\ \beta \geq 0$, then $(I + \mathcal{L})^{\beta/2}$ is an isomorphism of $\Lambda_{\alpha+\beta}^{p,q}$ and $\Lambda_\alpha^{p,q}$.

Before we prove our main theorem, we will introduce generalized Riesz potentials and some related theorems. The details are in Folland [3].

NOTATION. A distribution which is C^∞ away from 0 and homogeneous of degree $\alpha - Q$ is called a kernel of type α.

PROPOSITION 3 (FOLLAND[3]). *Suppose $0 < Re\,\alpha < Q$. The integral*

$$R_\alpha(x) = \frac{1}{\Gamma(\frac{\alpha}{2})} \int_0^\infty t^{\frac{\alpha}{2}-1} h(x,t)dt$$

converges absolutely for all $x \neq 0$, and R_α is a kernel of type α.

PROPOSITION 4 (FOLLAND[3]). *Suppose $f \in L^p$ and the integral*

$$g(x) = f * R_\alpha(x) = \int f(xy^{-1})R_\alpha(y)dy \quad 0 < Re\,\alpha < Q$$

converges absolutely for almost every x.

If $f \in \mathrm{Dom}(\mathcal{L}^{-\frac{\alpha}{2}})$, then $g \in L^p$ and $\mathcal{L}^{-\alpha/2}f = g$, i.e., $\mathcal{L}^{-\frac{\alpha}{2}}f = f * R_\alpha$

PROPOSITION 5 (FOLLAND [3]). *Suppose* $0 < \alpha < Q$, $1 < p < Q/\alpha$, *and* $\alpha = Q(\frac{1}{p} - \frac{1}{q})$, *and let* K *be a kernel of type* α. *If* $f \in L^p$, *then* $f * K$ *and* $K * f$ *exist a.e., and are in* L^q, *and there is a constant* C_p *such that*

$$\|f * K\|_q \leq C_p \|f\|_p \quad \text{and} \quad \|K * f\|_q \leq C_p \|f\|_p.$$

Now we are ready to prove our main theorem.

THEOREM 2.

$$\Lambda_{\alpha_1}^{p_1,q} \in \Lambda_{\alpha_2}^{p_2,q} \quad \textit{for} \quad \alpha_1 - \alpha_2 = Q(\frac{1}{p_1} - \frac{1}{p_2}) \geq 0.$$

Proof. Suppose $f \in \Lambda_{\alpha_1}^{p_1,q}$. By proposition [4] we can express f as

$$f = (\mathcal{L}^{(\alpha_1 - \alpha_2)/2} f) * R_{\alpha_1 - \alpha_2}.$$

Let's denote $\tilde{f} = (I + \mathcal{L})^{\alpha_1 - \alpha_2} f * R_{\alpha_1 - \alpha_2}$. Then

$$(I + \mathcal{L})^{(\alpha_1 - \alpha_2)/2} f \in \Lambda_{\alpha_2}^{p_1,q} \text{ by proposition [1].}$$

Since $R_{\alpha_1 - \alpha_2} \in L^1$ is also a kernel of type $\alpha_1 - \alpha_2 = Q(\frac{1}{p_1} - \frac{1}{p_2})$,

$$\tilde{f} \in L^{p_2}, \ \|\tilde{f}\|_{L^{p_2}} \leq C \|(I + \mathcal{L})^{(\alpha_1 - \alpha_2)/2} f\|_{L^{p_1}}$$

by proposition [4]. But since two norms $\|f\|_{L^{p_2}}$, $\|\tilde{f}\|_{L^{p_2}}$ are equivalent, we have shown that $f \in L^{p_2}$. Also by proposition[1], $(I + \mathcal{L})^{\frac{\alpha_1 - \alpha_2}{2}}$ gives rise to an isomorphism between

$$\Lambda_{\alpha_1}^{p_1,q} \quad \text{and} \quad \Lambda_{\alpha_2}^{p_1,q}.$$

Also the two norms $\|f\|_{\Lambda_{\alpha_1}^{p_1,q}}$, $\|(I + \mathcal{L})^{\frac{\alpha_1 - \alpha_2}{2}} f\|_{\Lambda_{\alpha_2}^{p_1,q}}$ are equivalent.

Now we want to prove $f \in \Lambda_{\alpha_2}^{p_2,q}$, i.e., have to show the following integral is finite.

$$\text{Int} = \int \frac{\|f(uv) - f(v)\|_{p_2}^q}{|u|^{\alpha_2 + Q}} du.$$

Using the above expression $\tilde{f} = (I+\mathcal{L})^{\frac{\alpha_1-\alpha_2}{2}} f * R_{\alpha_1-\alpha_2}$,

$$
\begin{aligned}
&\|f\|_{\Lambda^{p_2,q}_{\alpha_2}} \\
&\le C\|\tilde{f}\|_{\Lambda^{p_2,q}_{\alpha_2}} \\
&= \|(I+\mathcal{L})^{\frac{\alpha_1-\alpha_2}{2}} f * R_{\alpha_1-\alpha_2}\|_{\Lambda^{p_2,q}_{\alpha_2}} \\
&= \int \frac{\|(I+\mathcal{L})^{\frac{\alpha_1-\alpha_2}{2}} f * R_{\alpha_1-\alpha_2}(uv) - (I+\mathcal{L})^{\frac{\alpha_1-\alpha_2}{2}} f * R_{\alpha_1-\alpha_2}(v)\|^q_{p_2}}{|u|^{\alpha_2 q+Q}} du \\
&\le C \int \frac{\|(I+\mathcal{L})^{\frac{\alpha_1-\alpha_2}{2}} f(uv) - (I+\mathcal{L})^{\frac{\alpha_1-\alpha_2}{2}} f(v)\|^q_{p_1}}{|u|^{\alpha_2 q+Q}} du \\
&= C\|(I+\mathcal{L})^{\frac{\alpha_1-\alpha_2}{2}} f\|_{\Lambda^{p_1,q}_{\alpha_2}} \\
&\le C\|f\|_{\Lambda^{p_1,q}_{\alpha_1}}.
\end{aligned}
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References

1. *Space* (I), J. Math. Mech. **13** (1964), 407–479.
2. Y-M. Kim, *A Restriction theorem on Follanf-Stein Spaces*, J. Korean Math. Soc. **30** (1993), 25–39.
3. G. B. Folland, *Subelliptic estimates and functions on nilpotent Lie groups*, Ark. Mat. **13** (1974), 160–208.
4. H. Komatsu, *Fractional powers of operators*, Pacific. J. Math., **19** (1966), 285–474.
5. N. Aronszajn and K. T. Smith, *Thoery of Bessel potentials*, Part I, Ann. Inst. Fourier **11** (1961), 385–474.
6. E. M. Stein, *Singular integrals and Differentiability properties of functions*, Princeton University Press, Princeton, N.J., 1970.

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