Comm. Korean Math. Soc. 9 (1994), No. 1, pp. 61–66

ON MUTATIONS OF PRIME ASSOCIATIVE ALGEBRAS

YOUNGSO KO

1. Introduction

Let B be an algebra with multiplication denoted by xy over a field F of characteristic $\neq 2$. Associated with B are an anticommutative algebra B^- and a commutative algebra B^+ which are defined on the same vector space as B but with multiplications respectively given by anticommutative product $[x,y] = xy - yx$ and commutative product $x \circ y = \frac{1}{2}(xy + yx)$. Two well known anticommutative and commutative algebras are Lie and Jordan algebras which have a long standing history of applications to physics.

An algebra B is called *Lie-admissible* if the associated algebra B^- is a Lie algebra; that is B^- satisfies the Jacobi identity

$$\big[[x,y],z\big] + \big[[y,z],x\big] + \big[[z,x],y\big] = 0.$$

Similarly, B is termed *Jordan-admissible* if B^+ becomes a Jordan algebra; that is

$$\big[(x \circ x) \circ y\big] \circ x = (x \circ x) \circ (y \circ x).$$

The well-known examples of Lie-admissible algebras are the associative algebras and Lie algebras, and the associative algebras and Jordan algebras are Jordan-admissible.

Let A be an associative algebra with multiplication xy over a field F. Let p, q be two fixed elements in A, and let $A(p,q)$ denote the algebra with multiplication

$$x * y = xpy - yqx$$

defined on the vector space A. The resulting algebra has been called the (p,q)-*mutation* of A. For a fixed element $r \in A$, denote $A^{(r)}$ to be

Received June 12, 1992. Revised August 30, 1993.
Supported in part by the Ministry of Education, grant no. BSRIP 93-104.

the algebra with multiplication $x \underset{(r)}{\cdot} y = xry$, but with the same vector space as A. The algebra $A^{(r)}$ is called the *r-homotope* of A and it is clear that $A^{(r)}$ is also associative. A straightforward calculation shows that $A(p,q)^- = \{A^{(p+q)}\}^-$ and $A(p,q)^+ = \{A^{(p-q)}\}^+$. Hence any mutation algebra of an associative algebra is also Lie and Jordan-admissible. The structure of $A(p,q)$ has been investigated by a number of authors [4, 5, 6].

By a derivation of A we mean a linear map $\delta : A \to A$ satisfying $\delta(xy) = x\delta(y) + \delta(x)y$, and denote by $\operatorname{Der} A$ the set of derivations of A.

An algebra B is called *prime* if the product of any two nonzero ideals of B is nonzero. It can be shown that B is prime if and only if $xBy = (0)$ implies $x = 0$ or $y = 0$ [3].

In this paper we discuss derivations of $A(p,q)$ when $A(p,q)$ is prime and $p, q \in A$ satisfy the condition $A = Ap + Aq + R$ where R is a subspace of $Z(A)$. Also we investigate isomorphisms of $A(p,q)$ to $B(a,b)$ when $B(a,b)$ is prime and $p, q \in A$ satisfy the above condition.

2. Derivations of mutation algebras

The derivations of $A(p,q)$ have been studied in [2], for the case where $A(p,q)$ is prime and $p, q \in A$ satisfy the condition $A = Ap + Aq$. We discuss the derivations of $A(p,q)$ when $p, q \in A$ satisfy the condition $A = Ap + Aq + R$, where R is a subspace contained in the center $Z(A)$ of A. First, investigating derivations of $A^{(p)}$, we have the following:

THEOREM 1. *Let A be a prime associative algebra with 1 over F of characteristic $\neq 2$, and p be a fixed element of A satisfying the condition $A = Ap + R$, where R is a subspace of the center $Z(A)$ of A. Then a linear map $\delta : A \to A$ with $\delta(1) = \delta(p) = 0$, is a derivation of $A^{(p)}$ if and only if δ is a derivation of A.*

Proof. Suppose that $\delta \in \operatorname{Der} A$ with $\delta(p) = 0$, then

$$\delta(xpy) = \delta(x)py + x\delta(p)y + xp\delta(y) = \delta(x)py + xp\delta(y),$$

and this means that $\delta \in \operatorname{Der} A^{(p)}$.

Conversely, assume that $\delta \in \operatorname{Der} A^{(p)}$ with $\delta(p) = \delta(1) = 0$. For $x, y \in A$, let $x = up + r$ where $r \in R$. Then $\delta(x) = \delta(up1) + \delta(r) = \delta(u)p + \delta(r)$

and $\delta(r)$ satisfies $[\delta(r), p] = 0$. Thus

$$\delta(xy) = \delta(upy + ry) = \delta(u)py + up\delta(y) + \delta(ry). \tag{2.1}$$

Since

$$p\delta(ry) = 1p\delta(ry) = \delta(pry) = \delta(rpy) \quad \text{and}$$
$$p\{\delta(ry) - \delta(r)y - r\delta(y)\} = \delta(rpy) - \delta(r)py - rp\delta(y) = 0,$$

for any $z = vp + s \in A$, where $s \in R$, we have

$$\begin{aligned} pz\{\delta(ry) - \delta(r)y - r\delta(y)\} &= p(vp + s)\{\delta(ry) - \delta(r)y - r\delta(y)\} \\ &= pvp\{\delta(ry) - \delta(r)y - r\delta(y)\} \\ &\quad + sp\{\delta(ry) - \delta(r)y - r\delta(y)\} = 0. \end{aligned}$$

Thus $pA\{\delta(ry) - \delta(r)y - r\delta(y)\} = 0$. We may assume $p \neq 0$, which, by primeness of A, implies $\delta(ry) = \delta(r)y + r\delta(y)$. Hence from (2.1)

$$\begin{aligned} \delta(xy) &= \delta(u)py + up\delta(y) + \delta(r)y + r\delta(y) \\ &= \delta(x)y + x\delta(y), \end{aligned}$$

which means $\delta \in \operatorname{Der} A$.

For $x \in A$, let R_x denote the right multiplication in A by x. For the case where p satisfies the condition $A = Ap$, we have:

THEOREM 2. *Let A be a prime associative algebra with 1 and p be a fixed element of A satisfying the condition $A = Ap$. Let $\delta : A \to A$ be a linear map. Then $\delta \in \operatorname{Der} A^{(p)}$ if and only if $\delta - R_{\delta(1)} \in \operatorname{Der} A$ and $\delta(p) = \delta(1)p + p\delta(1)$.*

Proof. Suppose $\delta \in \operatorname{Der} A^{(p)}$ and $x = up$, then $\delta(x) = \delta(up1) = \delta(u)p + up\delta(1) = \delta(u)p + x\delta(1)$. Hence $(\delta - R_{\delta(1)})(xy) = \delta(upy) - xy\delta(1) = \delta(u)py + up\delta(y) - xy\delta(1) = (\delta(x) - x\delta(1))y + x\delta(y) - xy\delta(1) = (\delta - R_{\delta(1)})(x)y + x(\delta - R_{\delta(1)})(y)$. Thus $\delta - R_{\delta(1)} \in \operatorname{Der} A$, and clearly $\delta(p) = \delta(1p1) = \delta(1)p + p\delta(1)$.

Conversely, if $\delta - R_{\delta(1)} \in \operatorname{Der} A$ and $\delta(p) = \delta(1)p + p\delta(1)$, then $(\delta - R_{\delta(1)})(p) = \delta(p) - p\delta(1) = \delta(1)p$. Hence $\delta(xpy) = (\delta - R_{\delta(1)})(xpy) + xpy\delta(1) = (\delta - R_{\delta(1)})(x)py + x(\delta - R_{\delta(1)})(p)y + xp(\delta - R_{\delta(1)})(y) + xpy\delta(1) = \delta(x)py - x\delta(1)py + x\delta(1)py + xp\delta(y) = \delta(x)py + xp\delta(y)$.

In [2], it is proved that if A is not commutative and $A(p,q)$ is prime for $p \neq q$, then $\delta \in \operatorname{Der} A(p,q)$ implies $\delta \in \operatorname{Der} A^{(p)}$ and $\delta \in \operatorname{Der} A^{(q)}$. From this result we have:

THEOREM 3. *Assume that A is an associative not commutative algebra with 1 and $p \neq q$ are fixed elements of A such that $A = Ap + Aq + R$ where $R \subset Z(A)$ and $A(p,q)$ is prime.*

Then a linear map $\delta : A \to A$ with $\delta(p) = \delta(q) = \delta(1) = 0$ is a derivation of $A(p,q)$ if and only if δ is a derivation of A.

Proof. Suppose that $\delta \in \operatorname{Der} A(p,q)$. Since $A(p,q)$ is prime and A is not commutative, from [2; Proposition 5.3] $\delta \in \operatorname{Der} A^{(p)}$ and $\delta \in \operatorname{Der} A^{(q)}$.

Let $x, y \in A$ and $x = up + vq + r$, $r \in R$. Then

$$(2.2) \qquad \begin{aligned} \delta(xy) &= \delta\{(up + vq + r)y\} \\ &= \delta(u)py + up\delta(y) + \delta(v)qy + vq\delta(y) + \delta(ry). \end{aligned}$$

Since $\delta \in \operatorname{Der} A^{(p)}$ and $\delta \in \operatorname{Der} A^{(q)}$, we have $p\{\delta(ry) - \delta(r)y - r\delta(y)\} = 0$ and $q\{\delta(ry) - \delta(r)y - r\delta(y)\} = 0$. Hence for $z = w_1 p + w_2 q + t$, $t \in R$, $pz\{\delta(ry) - \delta(r)y - r\delta(y)\} = 0$. That is, $pA\{\delta(ry) - \delta(r)y - r\delta(y)\} = 0$. Primeness of $A(p,q)$ implies that A is prime [1; Theorem 2.5]. Assuming that $p \neq 0$, we get $\delta(ry) = \delta(r)y + r\delta(y)$. Hence from (2.2)

$$\begin{aligned} \delta(xy) &= \delta(u)py + \delta(v)qy + \delta(r)y + up\delta(y) + vq\delta(y) + r\delta(y) \\ &= \delta(x)y + x\delta(y), \end{aligned}$$

and therefore, $\delta \in \operatorname{Der} A$.

Conversely, if $\delta \in \operatorname{Der} A$ with $\delta(p) = \delta(q) = 0$, then $\delta \in \operatorname{Der} A^{(p)}$ and $\delta \in \operatorname{Der} A^{(q)}$, and hence $\delta \in \operatorname{Der} A(p,q)$.

If p, q be invertible in A, then $A = Ap + Aq$ and Osborn showed that $A(p,q)$ is prime if and only if A is prime [7; Theorem 3.1]. Hence we have:

COROLLARY 4. *Assume that A is a non-commutative prime associative algebra with 1 and $p \neq q$ are invertible in A. Then a linear map $\delta : A \to A$ with $\delta(p) = \delta(q) = \delta(1) = 0$ is a derivation of $A(p,q)$ if and only if δ is a derivation of A.*

For the case when $p, q \in A$ satisfy the condition $A = Ap + Aq$ and $A(p,q)$ is prime, it was proved that $\delta \in \operatorname{Der} A(p,q)$ is expressed as $\delta = \delta_0 + R_a$ for some $\delta_0 \in \operatorname{Der} A$ and $a = \delta(1)$ [2; Theorem 5.1].

3. Isomorphisms of mutation algebras

Isomorphisms of $A(p,q)$ to $B(a,b)$ were determined in terms of isomorphisms or anti-isomorphisms of A to B when B is prime with 1 and (a,b) is normal in B, that is $B = aB + bB = Ba + Bb$. In this section, we investigate a similar problem under the conditions that $A = Ap + Aq + R$ and $B(a,b)$ is prime, where R is a subspace of $Z(A)$. The following is our main result in this section.

THEOREM 5. *Assume that A, B are associative algebras with unity 1 and $p, q \in A$, $(p \neq q)$, $a, b \in B$, $(a \neq b)$, are fixed elements such that $A = Ap + Aq + R$ where $R \subset Z(A)$ and $B(a,b)$ is prime. If $\varphi : A(p,q) \to B(a,b)$ is an isomorphism with $\varphi(1) = 1$, then φ is an isomorphism or an anti-isomorphism of A to B.*

Proof. If B is not commutative, then by [2; Theorem 3.2], φ satisfies

$$(3.1) \quad \varphi(xpy) = \varphi(x)a\varphi(y) \text{ and } \varphi(xqy) = \varphi(x)b\varphi(y) \text{ for all } x, y \in A,$$

or

$$(3.2) \quad \varphi(xpy) = -\varphi(y)b\varphi(x) \text{ and } \varphi(xqy) = -\varphi(y)a\varphi(x) \text{ for all } x, y \in A.$$

If $x = up + vq + r$ with $r \in R$, then $\varphi(x) = \varphi(u)a + \varphi(v)b + \varphi(r)$. Assume φ satisfies (3.1). Then for any $y \in A$

$$(3.3) \quad \begin{aligned} \varphi(xy) &= \varphi(upy + vqy + ry) \\ &= \varphi(u)a\varphi(y) + \varphi(v)b\varphi(y) + \varphi(ry). \end{aligned}$$

Since $[\varphi(r), a] = 0$ and $[\varphi(r), b] = 0$, $a\{\varphi(ry) - \varphi(r)\varphi(y)\} = \varphi(rpy) - \varphi(r)a\varphi(y) = 0$ and $b\{\varphi(ry) - \varphi(r)\varphi(y)\} = 0$. For $z \in B$, then there exists $c = wp + sq + t$, $t \in R$, with $\varphi(c) = z$. Hence $az\{\varphi(ry) - \varphi(r)\varphi(y)\} = a\{\varphi(w)a + \varphi(s)b + \varphi(t)\}\{\varphi(ry) - \varphi(r)\varphi(y)\} = 0$. Primeness of $B(a,b)$ means that B is prime, and assuming that $a \neq 0$, we obtain $\varphi(ry) = \varphi(r)\varphi(y)$. From (3.3), it follows that $\varphi(xy) = \varphi(x)\varphi(y)$.

If φ satisfy (3.2) then

$$(3.4) \quad \begin{aligned} \varphi(xy) &= \varphi(upy + vqy + ry) \\ &= -\varphi(y)b\varphi(u) - \varphi(y)a\varphi(v) + \varphi(ry) \end{aligned}$$

From (3.2) $\{\varphi(ry)-\varphi(y)\varphi(r)\}b=-\varphi(pry)+\varphi(rpy)=0$ and $\{\varphi(ry)-\varphi(y)\varphi(r)\}a=-\varphi(rqy)+\varphi(rqy)=0$. Thus, for $z\in B$, $\{\varphi(ry)-\varphi(y)\varphi(r)\}za=\{\varphi(ry)-\varphi(y)\varphi(r)\}\{-b\varphi(w)-a\varphi(s)+\varphi(t)\}a=0$. Assuming that $a\neq 0$, $\varphi(ry)=\varphi(y)\varphi(r)$. By (3.4) we have $\varphi(xy)=\varphi(y)\varphi(x)$, which means φ is an anti-isomorphism of A to B.

Finally, if B is commutative, then by [2; Proposition 3.8] A is also commutative and φ is an isomorphism of $A^{(p-q)}$ to $B^{(a-b)}$. Thus for $x,y\in A$,

$$\begin{aligned}\varphi(x)\varphi(y)(a-b)=\varphi(x)(a-b)\varphi(y)&=\varphi(x(p-q)y)\\&=\varphi(xy(p-q)1)=\varphi(xy)(a-b).\end{aligned}$$

Because B is prime, commutative and $a-b\neq 0$, we have $\varphi(xy)=\varphi(x)\varphi(y)$ for all $x,y\in A$.

References

1. A. Eldugue and F. Montaner, *On mutations of associative algebras*, J. Korean Math. Soc. **28** (1991), 143–156.
2. A. Elduque, F. Montaner and H. C. Myung, *Automorphisms and derivations in a class of Jordan and Malcev admissible algebras*, Nova J. of Algebra and Geometry **2** (1993), to appear.
3. I. N. Herstein, *Topics in Ring Theory*, University of Chicago Press, 1969.
4. H. C. Myung, *The exponentiation and deformations of Lie-admissible algebras*, Hadronic J. **5** (1982), 771–903.
5. H. C. Myung and A. A. Sagle, *On Lie-admissible mutations of associative algebras*, Hadronic J. **10** (1987), 35–51.
6. R. H. Oehmke, *Some elementary structure theorems for a class of Lie-admissible algebras*, Hadronic J. **3** (1979), 293–319.
7. J. M. Osborn, *The Lie-admissible mutation $A(r,s)$ of an associative algebra A*, Hadronic J. **5** (1982), 904–930.

Department of Mathematics
Seoul National University
Seoul 151–742, Korea