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ASYMPTOTIC GRADE SCHEME AND GRADE FUNCTION ON MODULES

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1. Introduction

R will always be a commutative Noetherian ring, I will be an ideal of R and M will be a finitely generated R-module unless otherwise stated. We denote by $\mathbf{R}$ the Rees ring $R[u, It]$ of R with respect to I, where t is an indeterminate and $u = t^{-1}$. Similarly, we will denote by $\mathbf{M}$ the graded $\mathbf{R}$-module $M[u, It] = \bigoplus_{n\geq 0} I^n M$. It is finitely generated over $\mathbf{R}$. If (R, P) is a local ring, then R^* will be the P-adic completion of R and M^* will be the completion of M with respect to the P-adic filtration $\{P^n M\}_{n\geq 0}$. In particular, we will denote by R_P^* and M_P^* the P_P-adic completion of R_P and M_P, respectively, where R_P, P_P and M_P are localizations of R, P and M at P, respectively.

In 1976, L. J. Ratliff Jr. in [7], showed that if I is an ideal of a Noetherian ring R with $ht(I) \geq 1$ then the sets $Ass_R(R/(I^n)_a)$ are equal for all large n, where I_a is the integral closure of I and $Ass_R(M)$ is the set of all associate primes of M. Note that later he showed that the assumption that $ht(I) \geq 1$ is not necessary. We denote this set by $\bar{\mathbf{A}}^*(I, R)$, i.e., $\bar{\mathbf{A}}^*(I, R) = Ass_R(R/(I^n)_a) = Ass_R(R/(I^{n+1})_a) = \cdots$ for all large n. It was shown in [7] that $\bar{\mathbf{A}}^*(I, R) \subseteq \bigcup_{n=1}^{\infty} Ass_R(R/I^n)$. The question of when the sets $Ass_R(R/I^n)$ are equal for all large n was left open. This was settled in 1979, by M.Brodmann, in [2], even without the assumption that $ht(I) \geq 1$. He showed that if I is an ideal of a Noetherian ring R and M is a finitely generated R-module, then the sequences $\{Ass_R(M/I^n M)\}_{n\in N}$ and $\{Ass_R(I^n M/I^{n+1} M)\}_{n\in N}$ of sets of prime ideals of R are eventually constant. We denote these sets by $\mathbf{A}^*(I, M)$ and $\mathbf{B}^*(I, M)$, respectively. He also showed that $\mathbf{A}^*(I, M) \neq \mathbf{B}^*(I, M)$.

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It is also known that neither $\mathbf{A}^*(I, M)$ nor $\mathbf{B}^*(I, M)$ is monotone. For counterexamples, see [2] and [[4], page 2].

Since then these results have led to a large body of recent research. In particular, the author, in [1], studied various questions related to sets of prime ideals associated to powers of an ideal and a module over a Noetherian ring. It was shown that the relationship of the sets $\bar{\mathbf{A}}^*(I, M), \mathbf{A}^*(I, M), \bar{\mathbf{Q}}^*(I, M), \mathbf{Q}(I, M)$ and $\mathbf{E}(I, M)$ for rings have valid analogues for modules and that all these sets are finite.

On the other hand, S. McAdam, in [3], introduced and studied grade schemes and grade functions in the abstract. He gave a characterization of grade functions. In the same paper, he showed the existence of the canonical grade scheme for a certain grade function.

In section 2, we give a generalization of the grade schemes and grade functions for rings to essentially finite modules. Most of these results concerning such functions for rings which were shown in [3] also hold for essentially finite R-modules. Numerous such results could be mentioned here, but instead we refer the reader to [3]. However, we state a characterization of of grade functions.

In section 3, we define counterparts for modules of asymptotic (resp. essential) sequences, and asymptotic (resp. essential) grades. In theorems 3.9 and 3.10, it is shown that all maximal asymptotic (resp. essential) sequences have the same length. Finally, in theorem 3.13, we also give an explicit expression of the canonical grade scheme of asymptotic grade functions on modules.

2. Grade Schemes and Grade Functions

In this section, we give a generalization of the grade schemes and grade functions for rings to essentially finite modules. In theorem 2.10, we give a characterization of grade functions. All the proofs given in [3] can be easily carried over to modules, so that we omit them.

We begin this section by defining essentially finite modules.

DEFINITION 2.1. Let R be a Noetherian ring. An R-module M is said to be essentially finite if M is a finitely generated R_S-module for some multiplicatively closed subset S of R.

We note that every finitely generated R-module is an essentially finite R-module. We will denote by $\mathcal{M}$ a set of all essentially finite R-modules such that if $M \in \mathcal{M}$ and T is a multiplicative subset of R then $M_T \in \mathcal{M}$. Let $\mathcal{I}$ be the set of all ideals of R.

We have the following easy observations concerning the above definition.

LEMMA 2.2. *Let R be a Noetherian ring and M an essentially finite module; Say M is a finite R_S–module for some multiplicatively closed subset S of R.*

1. *For any $s \in S$, the homomorphism $\phi_s : M \longrightarrow M$ defined by $\phi_s(x) = sx$, for any $x \in M$, is a bijection.*
2. *If $IM \neq M$ then M/IM is S–torsion free and $I \bigcap S = \emptyset$.*
3. *If $P \in Supp(M)$ and $P \bigcap S = \emptyset$ then $PM_P \neq M_P$.*

Proof. This is straight forward.

DEFINITION 2.3. A function $A_R : \mathcal{I} \times \mathcal{M} \longrightarrow Spec(R)$ is said to be a proto–grade scheme on $\mathcal{M}$ if, for each $(I, M) \in \mathcal{I} \times \mathcal{M}$,

1. $A_R(I, M)$ is a finite subset of $Supp(M/IM)$.
2. $A_R(I, M) = \emptyset$ if and only if $IM = M$.
3. If T is a multiplicatively closed subset of R then

$$A_R(I, M_T) = \{P \in A_R(I, M) : P \bigcap T = \emptyset\}.$$

In the case that $\mathcal{M}^M$ is the set of all localizations of some essentially finite R–module M, we will say that $A_R(\, , \,)$ is a proto–grade scheme on M instead $\mathcal{M}^M$. In this case, a proto–grade scheme $A_R(\, , \,)$ is a function from $\mathcal{I} \times \mathcal{M}^M$ into the class of all finite subsets of $Supp_R(M)$.

REMARK 2.4. If $A_R(\, , \,)$ is a proto–grade scheme on $\mathcal{M}$ and T is a multiplicatively closed subset of R, then we have an induced proto–grade scheme $A_{R_T}(\, , \,)$ on $\mathcal{M}_T = \{M \in \mathcal{M} : M \text{ is an } R_T\text{–module}\}$ defined by $A_{R_T}(I_T, M) = \{PR_T : P \in A_R(I, M)\}$.

LEMMA 2.5. *Let $M \in \mathcal{M}, I \in \mathcal{I}$ and T be a multiplicatively closed subset of R, then*

$$A_{R_T}(I_T, M_T) = \{PR_T : P \in A_R(I, M_T)\}.$$

Proof. It is clear that $\{PR_T : P \in A_R(I, M_T)\} \subseteq A_{R_T}(I_T, M_T)$. Let $Q \in A_{R_T}(I_T, M_T)$. Then there exists a prime $P \in A_R(I, M)$ such that $Q = PR_T$. It is clear that $P \bigcap T = \emptyset$. Hence $P \in A_R(I, M_T)$.

We will denote $A_R(I, M)$ by $A(I, M)$ and $Supp_R(M)$ by $Supp(M)$ to simplify the notations unless we want to emphasize its dependence on R.

DEFINITION 2.6. Let $A(\, , \,)$ be a proto–grade scheme on $\mathcal{M}$ and let $M \in \mathcal{M}$. A sequence of elements $x_1, \dots, x_n$ of R is said to be an avoiding sequence for A on M if

(1) $(x_1, \dots, x_n)M \neq M$.
(2) For each $i = 1, 2, \dots, n$,

$$x_i \notin \bigcup \{P : P \in A((x_1, \dots, x_{i-1})R, M)\}.$$

In definition 2.6, we will consider the empty sequence as an avoiding sequence for A and the ideal generated by the empty sequence is the zero ideal of R.

We now state definitions of grade schemes and grade functions which will be considered in this paper.

DEFINITION 2.7. Let A be proto–grade scheme on $\mathcal{M}$.

(1) Let $M \in \mathcal{M}$. Then an avoding sequence $x_1, \dots, x_n$ contained in I is said to be a maximal avoiding sequence for A in I on M if

$$I \subseteq \bigcup \{P : P \in A((x_1, \dots, x_n)R, M)\}.$$

(2) If, for each $(I, M) \in \mathcal{I} \times \mathcal{M}$, all maximal avoiding sequences for A in I on M have the same length then A is said to be a grade scheme on $\mathcal{M}$ and we will call an avoiding sequence for A on M an A–sequence on M.
(3) Let A be a grade scheme on $\mathcal{M}$ and let $f(I, M)$ be the length of a maximal A–sequence in I on M, for any $(I, M) \in \mathcal{I} \times \mathcal{M}$. Then $f(\, , \,)$ is said to be the grade function of A.
(4) If $f(\, , \,)$ is the grade function of some grade scheme on $\mathcal{M}$, then it is said to be a grade function on $\mathcal{M}$.

DEFINITION 2.8. Let A and B be grade schemes on $\mathcal{M}$. Then

We write $A \subseteq B$ if $A(I, M) \subseteq B(I, M)$ for any $(I, M) \in \mathcal{I} \times \mathcal{M}$.

We note that almost all the results regarding above definitions which were shown in [3] hold for modules. The proofs for modules are essentially the same as those for rings, so that we omit them. We refer the reader who are interested in to [3]. However, we now state the results which are considered in this paper.

DEFINITION 2.9. Let $Q \in Spec(R)$ and $\mathbf{U}$ be an infinite subset of $Spec(R)$ such that for each $P \in \mathbf{U}$, $Q \subseteq P$. If for any infinite subset $\mathbf{V}$ of $\mathbf{U}$, we have $\bigcap\{P : P \in \mathbf{V}\} = Q$ then $(Q, \mathbf{U})$ is said to be a conforming pair.

DEFINITION 2.10. For a given nonnegative integer valued function f, we say that f satisfies condition (#) if f satisfies the following conditions

(1) $f(I, M) = \min\{f(P, M_P) : P \in Supp(M/IM)\}$.
(2) $f(P, M_P) \leq \dim(M_P)$ for all $P \in Supp(M)$.
(3) If $(Q, \mathbf{U})$ is a conforming pair and if $f(P, M_P) \leq n$ for all $P \in \mathbf{U}$ then $f(Q, M_Q) \leq n - 1$.

THEOREM 2.11 (CHARACTERIZATION OF GRADE FUNCTIONS). *Let $M \in \mathcal{M}$ and let f be a nononegative integer valued function defined on $\mathcal{I} \times \mathcal{M}$. Then f is a grade function on $\mathcal{M}$ if and only ifit satisfies the condition (#) . In this case, $A_f(I, M) = \{P \in Supp(M/IM) : f(I, M_P) = f(P, M_P)\}$ defined on $\mathcal{I} \times \mathcal{M}$ is a grade scheme on $\mathcal{M}$.*

Proof. This is similar to the proof of [[3], theorem(2.4)].

DEFINITION 2.12. If f be a grade function on $\mathcal{M}$ then A_f is said to be the canonical grade scheme of f on $\mathcal{M}$.

3. Asymptotic and Essential Grades on Modules

In this section, we define asymptotic (resp.essential) sequences, and asymptotic (resp. essential) grades on modules. we note that all results concerning asymptotic (resp. essential) sequences, and asymptotic (resp. essential) grades for rings have valid analogues for modules. This is expected from propositions 3.3 and 3.4. Numerous other results could be mentioned here but instead we refer to [1].

DEFINITION 3.1. Let R be a commutative noetherian ring with unity and I an ideal of R Let M be a finitely generated R-module.

(1) $\bar{\mathbf{Q}}^*(I,M) = \{P \in Spec(R)$; there exist a prime q minimal in $Supp_{R^*_P}(M^*_P)$ with PR^*_P minimal over $IR^*_P + q\}$
(2) $\bar{\mathbf{A}}^*(I,M) = \{P : P = Q \cap R$ for some $Q \in \bar{\mathbf{Q}}^*(u\mathbf{R},\mathbf{M})\}$
(3) $\mathbf{Q}(I,M) = \{P \in Spec(R)$: there exists a prime $q \in Ass_{R^*_P}(M^*_P)$ with P^*_P minimal over $IR^*_P + q\}$
(4) $\mathbf{E}(I,M) = \{P : P = Q \cap R$ for some $Q \in \mathbf{Q}(u\mathbf{R},\mathbf{M})\}$

LEMMA 3.2. *Let $A(I,M)$ represent any one of the above. Let S be a multiplicatively closed subset of R disjoint from the prime P. Then*

$$P \in A(I,M) \text{ if and only if } P_S \in A(I_S, M_S)$$

Proof. This is straight forward.

The following propositions were proved in [1]. We refer to [1] for a proof.

PROPOSITION 3.3. *Let R be a commutative noetherian ring with unity and I an ideal of R Let M be a finitely generated R-module.*

(1) *$P \in \mathbf{Q}(I,M)$ (resp. $\mathbf{E}(I,M)$) if and only if there exists $q \in Ass_R(M)$ with $q \subseteq P$ and $P/q \in \mathbf{Q}(I+q/q, R/q)$ (resp. $\mathbf{E}(I+q/q, R/q)$).*
(2) *$P \in \bar{\mathbf{Q}}^*(I,M)$ (resp. $\bar{\mathbf{A}}^*(I,M)$) if and only if there exists a prime q minimal in $Supp_R(M)$ with $q \subseteq P$ and*

$$P/q \in \bar{\mathbf{Q}}^*(I+q/q, R/q)(\text{resp. } \bar{\mathbf{A}}^*(I+q/q, R/q)).$$

PROPOSITION 3.4. *Let $A(I,M)$ denote any of $\bar{\mathbf{Q}}^*(I,M)$, $\bar{\mathbf{A}}^*(I,M)$, $\mathbf{A}^*(I,M)$, $\mathbf{Q}(I,M)$ or $\mathbf{E}(I,M)$. Let $\Phi : R \to T$ be a faithfully flat ring homomorphism.*

(1) *If $Q \in A(IT, M \bigotimes_R T)$, then $Q \bigcap R \in A^*(I,M)$.*
(2) *If $P \in A(I,M)$ and Q is a minimal prime ideal over PT, then $Q \in A(IT, M \bigotimes_R T)$.*

PROPOSITION 3.5. *Let R be a commutative noetherian ring with unity and I an ideal of R Let M be a finitely generated R-module. Then*

$$\mathbf{E}(I,M) \subseteq \mathbf{A}^*(I,M)$$

THEOREM 3.6. *Let R be a commutative noetherian ring with unity and I an ideal of R Let M be a finitely generated R-module. Then the following hold.*

(1) $\bar{\mathbf{Q}}^*(I,M) \subseteq \mathbf{Q}(I,M) \bigcap \bar{\mathbf{A}}^*(I,M)$
(2) $\mathbf{Q}(I,M) \bigcup \bar{\mathbf{A}}^*(I,M) \subseteq \mathbf{E}(I,M) \subseteq \mathbf{A}^*(I,M)$
(3) *The sets* $\bar{\mathbf{Q}}^*(I,M)$, $\bar{\mathbf{A}}^*(I,M), \mathbf{Q}(I,M)$ *and* $\mathbf{E}(I,M)$ *are finite.*

Proof. This follows from definition 3.1, proposition 3.3, 3.4 and 3.5 and those results known for rings.

We now define asymptotic (resp. essential) sequences on modules.

DEFINITION 3.7. The sequence of elements $x_1, x_2, \ldots, x_n$ of R is said to be an asymptotic (respectively, essential) sequence on M if

(1) $(x_1, \ldots, x_n)M \neq M$
(2) For $i = 1, \ldots, n x_i \notin \bigcup\{p : p \in \bar{\mathbf{A}}^*((x_1, \ldots, x_{i-1})R, M)\}$ (respectively, $x_i \notin \bigcup\{p : p \in \mathbf{E}((x_1, \ldots, x_{i-1})R, M)\}$)

LEMMA 3.8. *Let R be a local ring and M a finitely generated R-module. Let $x_1, \ldots, x_n \in R$ such that $(x_1, \ldots, x_n)M \neq M$.*

(1) *$x_1, \ldots, x_n$ is an asymptotic sequence on M if and only if for each minimal prime $q \in Supp(M^*)$, $(x_1, \ldots, x_n)R^* + q/q$ is an ideal of the principal class in R^*/q.*
(2) *$x_1, \ldots, x_n$ is an essential sequence on M if and only if for each prime $q \in Ass_{R^*}(M^*)$, $(x_1, \ldots, x_n)R^* + q/q$ is an ideal of the principal class in R^*/q.*

Proof. Let $x_1, \ldots, x_n$ be an asymptotic sequence on M. It follows from propositions 3.3 and 3.4 that $x_1 + q, \ldots, x_n + q$ is an asymptotic sequence on R^*/q. Hence $ht((x_1, \ldots, x_n)R^* + q/q) = n$. Conversely, by [lemma(5.3) in [4]], $x_1 + q, \ldots, x_n + q$ is an asymptotic sequence on R^*/q. Hence $x_1, \ldots, x_n$ is an asymptotic sequence on M.
A proof of (2) is analogous to the proof of (1)

The following two theorems show that all maximal asymptotic (resp. essential) sequences have the same length.

THEOREM 3.9. *Let R be a semi-local ring and M a finitely generated R-module. Then the following are equal.*

(1) *The length of any maximal asymptotic sequence in I on M.*
(2) $\min_P\{ht(IR_P^* + q/q); P \in Spec(R)$ *with* $I \subseteq P$ *and* q *a minimal prime in* $Supp(M_P^*)\}$.
(3) $\min_P\{\dim(M_P^*); P \in Spec(R)$ *with* $I \subseteq P\}$.

Proof. Let $x_1, \dots, x_n$ be a maximal asymptotic sequence in I on M. Let $P \in Spec(R)$ with $I \subseteq P$. Then for any minimal prime $q \in Supp(M_P^*)$ $ht((x_1, \dots, x_n)R_P^* + q/q) = n$. Hence $ht(IR_P^* + q/q) \geq n$. This shows that (No. in (2)) $\geq$ (No. in (1)). It is also clear that (No. in (3)) $\geq$ (No. in (2)). To complete the proof it suffices to show that (No. in (1)) $\geq$ (No. in (3)). Let $x_1, \dots, x_n$ be a maximal asymptotic sequence in I on M. Then there exists a prime ideal $Q \in \bar{\mathbf{A}}^*((x_1, \dots, x_n)R, M)$ such that $I \subseteq Q$. By proposition 3.4 and lemma 3.2 $Q_Q^* \in \bar{\mathbf{A}}^*((x_1, \dots, x_n)R_Q^*, M_Q^*)$. There exists a minimal prime $q \in Supp(M_Q^*)$ such that $Q_Q^*/q \in \bar{\mathbf{A}}^*((x_1, \dots, x_n)R_Q^* + q/q, R_Q^*/q)$. Since R_Q^*/q is quasi-unmixed local and $x_1 + q, \dots, x_n + q$ is an asymptotic sequence on R_Q^*/q, by [lemma(3.3) in [5]] $n = ht(Q_Q^*/q) = \dim(M_P^*)$. Therefore (No. in(3)) $\leq$ (No. in (1)).

THEOREM 3.10. *Let R be a semi-local ring and M a finitely generated R-module. Then the following are equal.*

(1) *The length of any maximal essential sequence in I on M.*
(2) $\min_P\{ht(IR_P^* + q/q)$; *for any* $P \in Spec(R)$ *with* $I \subseteq P$ *and for any prime* $q \in Ass_{R_P^*}(M_P^*)\}$.
(3) $\min_P\{\dim(R_P^*/q); P \in Spec(R)$ *with* $I \subseteq P$ *and a prime* $q \in Ass_{R_p^*}(M_p^*)\}$.

Proof. Let $x_1, \dots, x_n$ be a maximal essential sequence in I on M. Let $P \in Spec(R)$ with $I \subseteq P$. Then for any $q \in Ass_{R_P^*}(M_P^*)$ $ht((x_1, \dots, x_n)R_P^* + q/q) = n$. Hence $ht(IR_P^* + q/q) \geq n$ and this shows that (No. in (2))$\geq$(No. in (1)). It is also clear that (No. in (3))$\geq$(No. in (2)). To complete the proof it suffices to show that (No. in (1))$\geq$(No. in (3)). Let $x_1, \dots, x_n$ be a maximal essential sequence in I on M. Then there exists a prime ideal $Q \in \mathbf{E}((x_1, \dots, x_n)R, M)$ such that $I \subseteq Q$.

By proposition 3.4 and lemma 3.2 $Q_Q^* \in \mathbf{E}(x_1, \ldots, x_n)R_Q^*, M_Q^*)$. By proposition 3.3 there exists a prime ideal $q \in Ass_{R_Q^*}(M_Q^*)$ such that $Q_Q^*/q \in \mathbf{E}((x_1, \ldots, x_n)R_Q^* + q/q, R_Q^*/q)$. Since R_Q^*/q is unmixed local domain and $x_1 + q, \ldots, x_n + q$ is an essential sequence on R_Q^*/q, by [lemma(3.3) in [5]], $n = ht(Q_Q^*/q) = \dim(R_Q^*/q)$. Therefore (No. in (1))$\geq$(No. in (3)).

We are now ready to give the definitions of asymptotic and essential grades on modules.

DEFINITION 3.11. Let I be an ideal of R. The asymptotic (respectively, essential) grade of I on M is the length of a maximal asymptotic (respectively, essential) sequence in I on M and we will denote this by $agr(I, M)$ (respectively, $egr(I, M)$).

We showed in theorem 3.9 and 3.10 that all maximal asymptotic (resp. essential) sequences have the same length so that $agr(I, M)$ and $egr(I, M)$ are well defined.

LEMMA 3.12. *Let R and M be as in theorem 3.9 and let I be an ideal of R with $IM \neq M$. Then a regular sequence in I on M is an essential sequence in I on M and an essential sequence in I on M is an asymptotic sequence in I on M. In particular, $gr(I, M) \leq egr(I, M) \leq agr(I, M) \leq ht(I + q/q)$ for any minimal prime $q \in Supp(M)$.*

Proof. By theorem 3.6 $\bar{\mathbf{A}}^*(I, M) \subseteq \mathbf{E}(I, M) \subseteq \mathbf{A}^*(I, M)$. It is clear that $agr(I, M) \leq ht(I + q/q)$. Hence, the lemma follows from these.

We now closed this section with stating some examples of grade schemes and grade functions on modules and with identifying the canonical grade scheme A_f when f is the asymptotic grade function on $\mathcal{M}$. The essential case is analogous to the asymptotic case.

EXAMPLE. Let R be a Noetherian ring and M a finitely generated R-module.

(1) Let $A_1(I, M) = Ass_R(M/IM)$. Then $A_1(I, M)$ is the grade scheme on M, the A_1-sequences are regular sequences on M and the grade function $f_1(I, M)$ is classical grade.
(2) Let $A_2(I, M) = \mathbf{A}^*(I, M) = Ass_R(M/(I^n M))$ for all large n. It is known that in general, $Ass_R(M/IM)$ and $Ass_R(M/(I^n M))$

are not comparable. But if I is generated by a R–sequence then

$$Ass_R(M/IM) = Ass_R(M/(I^nM)) \text{ for any } n \geq 1.$$

In this case $A_1(I,M) = A_2(I,M)$ and $f_1(I,M) = f_2(I,M)$.

(3) Let $A_3(I,M) = \bar{\mathbf{A}}^*(I,M)$. Then A_3–sequences are asymptotic sequences on M and the grade function $f_3(I,M)$ is asymptotic grade.

(4) Let $A_4(I,M) = \mathbf{E}(I,M)$. Then A_4–sequences are essential sequences on M and the grade function $f_4(I,M)$ is essential grade.

(5) Let

$$A_5(I,M) = \{P \in Spec(R) : P \text{ is minimal in } Supp(M/IM)\}$$

Then $f_5(I,M) = ht((I + (0 :_R M))/(0 :_R M))$.

THEOREM 3.13. *Let M be a finitely generated R–module and let f be asymptotic grade function on M. Then $P \in A_f(I,M)$ if and only if, for each minimal prime $q \in Supp(M_P^*)$, $ht((IR_P^*+q)/q) \geq \min\{\dim(R_P^*/p) :$ p minimal in $Supp(M_P^*)\}$. In particular, $A_f(I,M) \subseteq \bar{\mathbf{A}}^*(I,M)$.*

Proof. By theorem 3.9, $f(P, M_P) = \min\{\dim(R_P^*/p) : p \text{ minimal in } Supp(M_P^*)\}$ and $f(I, M_P) = \min\{ht((IR_P^* + q)/q) : q \text{ minimal in } Supp(M_P^*)\}$. Hence the first assertion follows. To show that $A_f(I,M) \subseteq \bar{\mathbf{A}}^*(I,M)$, let q be a minimal prime in $Supp(M_P^*)$ such that $\dim(R_P^*/q) = f(P, M_P)$. If $P \in A_f(I,M)$, then by the above, $ht((IR_P^* + q)/q) = \dim(R_P^*/q) = ht(P_P^*/q)$. Hence P_P^* is minimal over $IR_P^* + q$. Hence $P \in \bar{\mathbf{A}}^*(I,M)$.

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