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PROPERTIES OF LATTICES PRESERVED BY TAKING RECTANGULAR PRODUCTS

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A lattice is called *bounded* if it has both the least element and the largest element which are usually denoted by 0 and 1, respectively. Recently, Bennett [4] defined the *rectangular product* of two bounded lattices L and M, denoted by $L \square M$, to be the set

$$\{(x,y) \mid x \in L, y \in M,\ x \neq 0, y \neq 0\} \cup \{(0,0)\}$$

with the order induced from the direct product $L \times M$, which is also a bounded lattice.

It follows immediately from the definition of the rectangular product that all joins and nonzero meets agree in both $L \square M$ and $L \times M$. Hence any equation which is satisfied by $L \times M$ will be satisfied in sublattices of $L \square M$ which do not contain $(0,0)$. Our intent here is to discuss some properties of given bounded lattices preserved under the rectangular product construction. We assume throughout this paper that all lattices are bounded.

Neither modularity nor distributivity is preserved by taking rectangular products. Here we list some of known results from Bennett [4]. We call a lattice *atomic* when every nonzero element is the join of the atoms below it.

THEOREM A. *Completeness and atomicity are preserved by taking rectangular product.*

A lattice in which each element has one and only one complement is called *uniquely complemented.*

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THEOREM B. *The rectangular product of two lattices is uniquely complemented if and only if one of the factors is uniquely complemented and the other is isomorphic to* $\mathbf{2}$.

An atomic lattice L is said to be *biatomic* if whenever a and b are nonzero elements and p is an atom in L with $p \leq a \vee b$, there are atoms $a_1 \leq a$ and $b_1 \leq b$ with $p \leq a_1 \vee b_1$.

THEOREM C. *The rectangular product of lattices* L_1 *and* L_2 *is biatomic if and only if both* L_1 *and* L_2 *are.*

We say that (a, b) is a *modular pair* in a lattice L if $x \leq b$ implies that $x \vee (a \wedge b) = (x \vee a) \wedge b$ for any $x \in L$. The lattice L is called *weakly modular* if $a \wedge b \neq 0$ implies that a and b form a modular pair for any $a, b \in L$.

THEOREM D. *If* L_1 *and* L_2 *are modular lattices, then* $L_1 \square L_2$ *is weakly modular. If* L_1 *and* L_2 *are weakly modular, then so is* $L_1 \square L_2$.

A lattice L is said to be have the *anti-exchange property* if $p \leq q \vee a$ and $p \not\leq a$ imply $q \not\leq p \vee a$ where p and q are distinct atoms.

THEOREM E. *If the rectangular product of atomic lattices* L_1 *and* L_2 *has the anti-exchange property, then so do* L_1 *and* L_2.

Let L be a lattice. For elements $a > b$ in L, we write $a \succ b$ or $b \prec a$ (a *covers* b or b is *covered by* a) if $a \geq c > b$ implies $a = c$ for every element c of L. An *atom* is any element which covers the least element and a *dual atom* is any element which is covered by the greatest element. Let us denote by $A(L)$ and $DA(L)$ the sets of all atoms and dual atoms, respectively, of L. Then $A(L_1 \square L_2) = A(L_1) \times A(L_2)$. The atoms of $L_1 \square L_2$ are exactly the elements (p, q), where p and q are atoms of L_1 and L_2, respectively. Furthermore, the dual atoms in $L_1 \square L_2$ are of the form $(c, 1)$ or $(1, d)$, where c and d are dual atoms of L_1 and L_2, respectively. If two dual atoms of the rectangular product have a zero meet, then they are of the same form.

REMARK. In [4], Bennett also proved the converse of Theorem E, but this is not true. In Figure 1, $\mathbf{2}^2$ is atomic and satisfies the anti-exchange property, but $\mathbf{2}^2 \square \mathbf{2}^2$ does not. In fact, for the atoms (a, d) and (b, d) in $\mathbf{2}^2 \square \mathbf{2}^2$ and the element (a, d) with $(a, d) \not\leq (1, c)$, we have $(a, d) \leq$

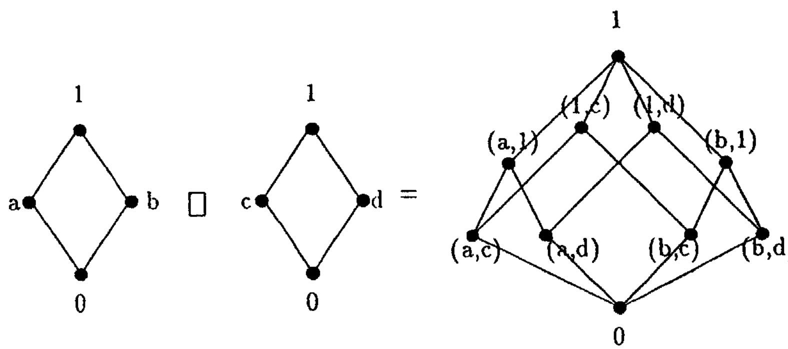

Figure 1.

$(b,d)\vee(1,c)=(1,1)$ in $\mathbf{2}^2\,\square\,\mathbf{2}^2$. But we have $(b,d)\leq(a,d)\vee(1,c)=(1,1)$ in $\mathbf{2}^2\,\square\,\mathbf{2}^2$.

The *length* of an n-element chain $\mathbf{n}$ is defined to be $n-1$. More generally, the *length* $l(P)$ of an ordered set P is defined as the supremum of the lengths of chains in P. In an ordered set P of finite length with the least element 0, the *height* $h(x)$ of an element $x\in P$ is $l([0,x])$. If P has the greatest element 1, then clearly $h(1)=l(P)$.

An ordered set P of finite length with 0 is called *graded* if for $x,y\in P$, $x\leq y$ and $h(x)+1=h(y)$ if and only if $x\prec y$, and is said to satisfy the *Jordan-Dedekind Chain Condition* if all maximal chains between the same endpoints have the same finite length. Observe that an ordered set P with 0 is graded if and only if every interval of P is of finite length and satisfies the Jordan-Dedekind Chain Condition.

We know that $h(a,b)=h(p,b)+l([(p,b),(a,b)])=h(b)+(h(a)-1)=h(a)+h(b)-1$ for some atom $p\leq a$ in a lattice L. Thus one has the following.

LEMMA 1. *Let L_1 and L_2 be lattices. For a nonzero element (a,b) in $L_1\,\square\,L_2$, we have $h(a,b)=h(a)+h(b)-1$.*

For lattices L_1 and L_2 with $(a,b)\neq(0,0)$, $(a,b)\prec(c,d)$ in $L_1\,\square\,L_2$ if and only if $a=c$ in L_1 and $b\prec d$ in L_2 or $a\prec c$ in L_1 and $b=d$ in L_2. Furthermore, $(0,0)$ is covered by the elements of the form (p,q), where both p and q are atoms of L_1 and L_2, respectively.

THEOREM 1. *The rectangular product of lattices L_1 and L_2 is graded if and only if both L_1 and L_2 are.*

Proof. Suppose that L_1 and L_2 are graded. It is enough to show that $(a,b) \leq (c,d)$ and $h(a,b)+1 = h(c,d)$ if and only if $(a,b) \prec (c,d)$ in $L_1 \mathbin{\square} L_2$. Suppose that $(a,b) \leq (c,d)$ in $L_1 \mathbin{\square} L_2$ and $h(a,b)+1 = h(c,d)$. If $(a,b) = (0,0)$ in $L_1 \mathbin{\square} L_2$, then we have $(a,b) = (0,0) \prec (c,d)$ in $L_1 \mathbin{\square} L_2$. If $(a,b) \neq (0,0)$ in $L_1 \mathbin{\square} L_2$, then $h(a)+h(b) = h(a,b)+1 = h(c,d) = h(c)+h(d)-1$ by Lemma 1. Since $h(a)$, $h(b)$, $h(c)$ and $h(d)$ are positive integers and $(a,b) \leq (c,d)$ in $L_1 \mathbin{\square} L_2$, $h(a)+1 = h(c)$ and $h(b) = h(d)$ or $h(a) = h(c)$ and $h(b)+1 = h(d)$. Since L_1 and L_2 are graded, $a \prec c$ in L_1 and $b = d$ in L_2 or $a = c$ in L_1 and $b \prec d$ in L_2 and so $(a,b) \prec (c,d)$ in $L_1 \mathbin{\square} L_2$. Conversely, suppose that $(a,b) \prec (c,d)$ in $L_1 \mathbin{\square} L_2$. If $a = 0$ in L_1 or $b = 0$ in L_2, then $(a,b) = (0,0) \prec (c,d)$ in $L_1 \mathbin{\square} L_2$, and hence $h(a,b)+1 = 1 = h(c,d)$. If $a \neq 0$ in L_1 and $b \neq 0$ in L_2, then we have $a \prec c$ in L_1 and $b = d$ in L_2 or $a = c$ and $b \prec d$ in L_2. Since L_1 and L_2 are graded, we have $h(a)+1 = h(c)$ and $h(b) = h(d)$ or $h(a) = h(c)$ and $h(b)+1 = h(d)$. Hence $(a,b) \leq (c,d)$ in $L_1 \mathbin{\square} L_2$ and $h(a,b)+1 = h(a)+h(b) = h(c)+h(d)-1 = h(c,d)$ by Lemma 1. Therefore $L_1 \mathbin{\square} L_2$ is graded.

Now, suppose that $L_1 \mathbin{\square} L_2$ is graded. Let $a \leq c$ in L_1 and $h(a)+1 = h(c)$. If $a = 0$ in L_1, then $h(a)+1 = 1 = h(c)$ and so c is an atom in L_1. Thus $a = 0 \prec c$ in L_1. If a is a nonzero element in L_1, then, for any nonzero element b in L_2, $(a,b) \leq (c,b)$ in $L_1 \mathbin{\square} L_2$ and $h(a,b)+1 = h(a)+h(b) = (h(c)-1)+h(b) = h(c,b)$ by Lemma 1. Since $L_1 \mathbin{\square} L_2$ is graded, $(a,b) \prec (c,b)$ in $L_1 \mathbin{\square} L_2$. Hence $a \prec c$ in L_1. Conversely, suppose that $a \prec c$ in L_1. If $a = 0$ in L_1, then c is an atom in L_1 and so $h(a)+1 = 1 = h(c)$. If $a \neq 0$ in L_1, then $a \leq c$ in L_1 and $(a,b) \prec (c,b)$ in $L_1 \mathbin{\square} L_2$ for any nonzero element b in L_2. Since $L_1 \mathbin{\square} L_2$ is graded, we have $h(a,b)+1 = h(c,b)$ and so $h(a,b) = h(a)+h(b)-1$ and $h(c,b) = h(c)+h(b)-1 = h(a,b)+1$ by Lemma 1. Hence $h(a)+1 = h(c)$. Thus L_1 is graded. Similarly, L_2 is also graded.

Let L be a complete lattice and let a be an element of L. Then a is called *compact* if $a \leq \bigvee X$ for some $X \subseteq L$ implies $a \leq \bigvee X_1$ for some finite $X_1 \subseteq X$. A complete lattice is called *algebraic* if every element is the join of compact elements.

THEOREM 2. *The rectangular product of lattices L_1 and L_2 is algebraic if and only if both L_1 and L_2 are.*

Proof. Note that L_1 and L_2 are complete if and only if $L_1 \,\square\, L_2$ is complete by Theorem A. Suppose that L_1 and L_2 are algebraic. Take any element (a, b) of $L_1 \,\square\, L_2$. Since L_1 and L_2 are algebraic, a is a join of compact elements p_i in L_1 and b is a join of compact elements q_j in L_2. Then $(a, b) = (\bigvee p_i, \bigvee q_j) = \bigvee_{i,j}(p_i, q_j)$. Now it is enough to show that (p_i, q_j) is compact in $L_1 \,\square\, L_2$.

Suppose that $(p_i, q_j) \leq \bigvee A$ for some subset A of $L_1 \,\square\, L_2$. Since $(p_i, q_j) \leq \bigvee A \leq \bigvee\{(x, y) \mid x_1 \in A_1 \text{and } x_2 \in A_2\}$ in $L_1 \square L_2$, where $A_1 = \{a \in L_1 \mid (a, b) \in A\}$ and $A_2 = \{b \in L_2 \mid (a, b) \in A\}$. Thus $p_i \leq \bigvee A_1$ in L_1 and $q_j \leq \bigvee A_2$ in L_2. Since p_i and q_j are compact elements in L_1 and L_2, respectively, there are finite subsets $B_1 \subseteq A_1$ and $B_2 \subseteq A_2$ such that $p_i \leq \bigvee B_1$ in L_1 and $q_j \leq \bigvee B_2$ in L_2. Hence $(p_i, q_j) \leq (\bigvee B_1, \bigvee B_2)$ in $L_1 \square L_2$. Let $B = \{(a, b) \mid a \in B_1,\, b \in B_2\}$. Thus $B \bigcap A$ is a finite subset of A and $(p_i, q_j) \leq \bigvee(B \bigcap A)$ and so (p_i, q_j) is compact in $L_1 \,\square\, L_2$.

Conversely, suppose that $L_1 \,\square\, L_2$ is algebraic and let a be an element of L_1. For each element b of L_2, $(a, b) = \bigvee(p_i, q_j) = (\bigvee p_i, \bigvee q_j)$ for some compact elements (p_i, q_j) of $L_1 \,\square\, L_2$. Thus $a = \bigvee p_i$ in L_1. We next show that p_i is a compact element in L_1. If $p_i \leq \bigvee X$ for some subset X of L_1, then $(p_i, q) \leq (\bigvee X, q)$ in $L_1 \,\square\, L_2$ for some element q of $\{q_j \mid (a, b) = \bigvee(p_i, q_j)\}$, and hence $(p_i, q) \leq (\bigvee X, q) = \bigvee(X, q)$. Since $L_1 \,\square\, L_2$ is an algebraic and (p_i, q) is a compact element in $L_1 \,\square\, L_2$, there is finite subset X_1 of X such that $(p_i, q) \leq \bigvee(X_1, q) = (\bigvee X_1, q)$ in $L_1 \,\square\, L_2$. Hence $p_i \leq \bigvee X_1$ for some finite subset X_1 of X. Thus p_i is a compact element in L_1. Therefore L_1 is algebraic. Similarly, L_2 is also algebraic.

A lattice L is called *semimodular* if it satisfies the *upper covering condition*, that is, $x \succ x \wedge y$ implies that $x \vee y \succ y$ in L.

THEOREM 3. *The rectangular product of lattices L_1 and L_2 is semimodular if and only if both L_1 and L_2 are semimodular and $\mid A(L_1) \mid = 1$ or $\mid A(L_2) \mid = 1$.*

Proof. Suppose that L_1 and L_2 are semimodular. We may assume without loss of generality that $\mid A(L_2) \mid = 1$. If $(a, b) \succ (a, b) \wedge (c, d)$ in

$L_1 \sqcap L_2$, then we have three cases to consider.

Case 1. $b \wedge d = 0$ in L_2.

Since $(a,b) \succ (0,0) = (a,b) \wedge (c,d)$ in $L_1 \sqcap L_2$, both a and b are atoms of L_1 and L_2, respectively. Since $|A(L_2)| = 1$ and $b \wedge d = 0$ in L_2, $d = 0$ in L_2. Either $a \wedge c = 0$ or $a \wedge c \neq 0$ in L_1, hence we have $(c,d) = (0,0)$ in $L_1 \sqcap L_2$. Thus $(a,b) \vee (c,d) = (a,b) \succ (0,0) = (c,d)$ in $L_1 \sqcap L_2$.

Case 2. $a \wedge c = 0$ in L_1 and $b \wedge d \neq 0$ in L_2.

Since $(a,b) \succ (0,0) = (a,b) \wedge (c,d)$ in $L_1 \sqcap L_2$, both a and b are atoms of L_1 and L_2, respectively, and hence $a \succ 0 = a \wedge c$ in L_1. Since $|A(L_2)| = 1$, we have $b \leq d$ in L_2. By the semimodularity of L_1, $a \vee c \succ c$ in L_1, and hence $(a,b) \vee (c,d) = (a \vee c, d) \succ (c,d)$ in $L_1 \sqcap L_2$.

Case 3. $a \wedge c \neq 0$ in L_1 and $b \wedge d \neq 0$ in L_2.

Since $(a,b) \succ (a,b) \wedge (c,d) = (a \wedge c, b \wedge d) \neq (0,0)$ in $L_1 \sqcap L_2$, either $a \succ a \wedge c$ in L_1 and $b = b \wedge d$ in L_2 or $a = a \wedge c$ in L_1 and $b \succ b \wedge d$ in L_2. If $a \succ a \wedge c$ in L_1 and $b = b \wedge d$ in L_2, then $b \leq d$ in L_2. By the semimodularity of $L_1, a \vee c \succ c$ in L_1. Hence $(a,b) \vee (c,d) = (a \vee c, b \vee d) = (a \vee c, d) \succ (c,d)$ in $L_1 \sqcap L_2$. If $a = a \wedge c$ in L_1 and $b \succ b \wedge d$ in L_2, then $a \leq c$ in L_1. By the semimodularity of L_2, and $b \vee d \succ d$ in L_2. Thus $(a,b) \vee (c,d) = (c, b \vee d) \succ (c,d)$ in $L_1 \sqcap L_2$.

Conversely, suppose that $L_1 \sqcap L_2$ is semimodular. We show that L_1 and L_2 are semimodular and that $|A(L_1)| = 1$ or $|A(L_2)| = 1$.

Suppose that $a \succ a \wedge c$ in L_1. If $a \wedge c = 0$ in L_1, then a is an atom in L_1. For any atom b in L_2, $(a,b) \succ (0,0) = (a,b) \wedge (c,b)$ in $L_1 \sqcap L_2$. Since $L_1 \sqcap L_2$ is semimodular, $(a,b) \vee (c,b) = (a \vee c, b) \succ (c,b)$ in $L_1 \sqcap L_2$, and hence $a \vee c \succ c$ in L_1. If $a \wedge c \neq 0$ in L_1, then $(a,b) \succ (a \wedge c, b) = (a,b) \wedge (c,b)$ for any nonzero element b in L_2. Since $L_1 \sqcap L_2$ is a semimodular, we have $(a,b) \vee (c,b) = (a \vee c, b) \succ (c,b)$ in $L_1 \sqcap L_2$. Hence $a \vee c \succ c$ in L_1. Thus L_1 is semimodular. Similarly, L_2 is also semimodular. We next show that $|A(L_1)| = 1$ or $|A(L_2)| = 1$. Suppose that $|A(L_1)| \geq 2$ and $|A(L_2)| \geq 2$. Then there exist two distinct atoms p_1, q_1 in L_1 and two distinct atoms p_2, q_2 in L_2. Since $(p_1,p_2) \succ (0,0) = (p_1,p_2) \wedge (q_1,q_2)$ in $L_1 \sqcap L_2$ and $L_1 \sqcap L_2$ is semimodular, we have $(p_1,p_2) \vee (q_1,q_2) \succ (q_1,q_2)$ in $L_1 \sqcap L_2$. But we have $(p_1,p_2) \vee (q_1,q_2) = (p_1 \vee q_1, p_2 \vee q_2) \succ (p_1 \vee q_1, p_2) \succ (p_1,p_2)$ in $L_1 \sqcap L_2$, which is a contradiction to the semimodularity of $L_1 \sqcap L_2$.

Hence $|A(L_1)| = 1$ or $|A(L_2)| = 1$.

A lattice L is said to be *join-semidistributive* if $a \vee b = a \vee c$ implies that $a \vee b = a \vee (b \wedge c)$ for all $a, b, c \in L$. An ordered set P is said to satisfy the *Ascending Chain Condition* (ACC) if every increasing chain terminates:i.e., if $x_0 \leq x_1 \leq \cdots \leq x_n \leq \cdots$ in P, then $x_m = x_{m+1} = \cdots$ for some positive integer m.

THEOREM 4. *Let L_1 and L_2 be lattices.*

(i) *If $L_1 \square L_2$ is join-semidistributive, then L_1 and L_2 are join-semidistributive.*

(ii) *If L_1 and L_2 are biatomic, join-semidistributive and satisfy the ACC, then $L_1 \square L_2$ is join-semidistributive.*

Proof. (i) Let $L_1 \square L_2$ be join-semidistributive. Suppose that $a \vee b = a \vee c$ in L_1 for $a, b, c \in L_1$. If $a = 0$ or $b = 0$ or $c = 0$ in L_1, then clearly $a \vee b = a \vee (b \wedge c)$. Now we assume that $a \neq 0$, $b \neq 0$ and $c \neq 0$ in L_1. Since $a \vee b = a \vee c$ in L_1, $(a \vee b, d) = (a \vee c, d)$ for any nonzero element d in L_2. Hence $(a \vee b, 1) = (a \vee c, 1)$ in $L_1 \square L_2$, that is, $(a, 1) \vee (b, 1) = (a, 1) \vee (c, 1)$ in $L_1 \square L_2$. Since $L_1 \square L_2$ is join-semidistributive, $(a, 1) \vee (b, 1) = (a, 1) \vee ((b, 1) \wedge (c, 1))$ in $L_1 \square L_2$, and hence

$$(a \vee b, 1) = \begin{cases} (a, 1) & \text{if } b \wedge c = 0 \text{ in } L_1, \\ (a \vee (b \wedge c), 1) & \text{if } b \wedge c \neq 0 \text{ in } L_1. \end{cases}$$

Thus, either $b \wedge c = 0$ in L_1 or $b \wedge c \neq 0$ in L_1, $a \vee b = a \vee (b \wedge c)$ in L_1. Hence L_1 is join-semidistributive. Similarly, L_2 is also join-semidistributive.

(ii) Let L_1 and L_2 be biatomic, join-semidistributive and satisfy the ACC. Suppose that $(a_1, a_2) \vee (b_1, b_2) = (a_1, a_2) \vee (c_1, c_2)$ in $L_1 \square L_2$ for any elements $(a_1, a_2), (b_1, b_2)$ and (c_1, c_2) in $L_1 \square L_2$. Thus $(a_1 \vee b_1, a_2 \vee b_2) = (a_1 \vee c_1, a_2 \vee c_2)$ in $L_1 \square L_2$, and hence $a_1 \vee b_1 = a_1 \vee c_1$ in L_1 and $a_2 \vee b_2 = a_2 \vee c_2$ in L_2. Since L_1 and L_2 are join-semidistributive lattices, $a_1 \vee b_1 = a_1 \vee (b_1 \wedge c_1)$ in L_1 and $a_2 \vee b_2 = a_2 \vee (b_2 \wedge c_2)$ in L_2.

Case 1. $b_1 \wedge c_1 \neq 0$ in L_1 and $b_2 \wedge c_2 \neq 0$ in L_2.

Since $b_1 \wedge c_1 \neq 0$ in L_1 and $b_2 \wedge c_2 \neq 0$ in L_2, we have, in $L_1 \square L_2$,

$$\begin{aligned}(a_1, a_2) \vee (b_1, b_2) &= (a_1 \vee b_1, a_2 \vee b_2) \\ &= (a_1 \vee (b_1 \wedge c_1), a_2 \vee (b_2 \wedge c_2))\end{aligned}$$

$$= (a_1, a_2) \vee ((b_1 \wedge c_1), (b_2 \wedge c_2))$$
$$= (a_1, a_2) \vee ((b_1, b_2) \wedge (c_1, c_2)).$$

Case 2. $b_1 \wedge c_1 = 0$ in L_1 or $b_2 \wedge c_2 = 0$ in L_2.

Since $(a_1, a_2) \vee ((b_1, b_2) \wedge (c_1, c_2)) = (a_1, a_2)$, it is enough to show that $(b_1, b_2) \leq (a_1, a_2)$ in $L_1 \square L_2$. Take any atom $(p, q) \leq (b_1, b_2)$ in $L_1 \square L_2$. Then $(p, q) \leq (a_1, a_2) \vee (b_1, b_2) = (a_1, a_2) \vee (c_1, c_2)$ $L_1 \square L_2$. Since $L_1 \square L_2$ is a biatomic, there is an atom $(r_{11}, r_{21}) \leq (c_1, c_2)$ in $L_1 \square L_2$ such that $(p, q) \leq (a_1, a_2) \vee (r_{11}, r_{21})$. Since $(r_{11}, r_{21}) \leq (c_1, c_2) \leq (c_1, c_2) \vee (a_1, a_2) = (a_1, a_2) \vee (b_1, b_2)$, there is an atom $(s_{11}, s_{21}) \leq (b_1, b_2)$ in $L_1 \square L_2$ such that $(p, q) \leq (a_1, a_2) \vee (r_{11}, r_{21}) \leq (a_1, a_2) \vee (s_{11}, s_{21})$. Continuing this process, we have $(p, q) \leq (a_1, a_2) \vee (r_{11}, r_{21}) \leq (a_1, a_2) \vee (s_{11}, s_{21}) \leq \cdots \leq (a_1, a_2) \vee (r_{1k}, r_{2k}) \leq (a_1, a_2) \vee (s_{1k}, s_{2k}) \leq \cdots$, where $(r_{1k}, r_{2k}) \leq (c_1, c_2)$ such that $(s_{1(k-1)}, s_{2(k-1)}) \leq (a_1, a_2) \vee (r_{1k}, r_{2k})$ and $(s_{1k}, s_{2k}) \leq (b_1, b_2)$ such that $(r_{1k}, r_{2k}) \leq (a_1, a_2) \vee (s_{1k}, s_{2k})$. Since L_1 and L_2 satisfy the ACC, and hence $L_1 \square L_2$ satisfies the ACC, we have $(a_1, a_2) \vee (r_{1k}, r_{2k}) = (a_1, a_2) \vee (s_{1k}, s_{2k})$. If $(r_{1k}, r_{2k}) = (s_{1k}, s_{2k})$ in $L_1 \square L_2$, then $(p, q) \leq (a_1, a_2) \vee (r_{1k}, r_{2k}) = (a_1, a_2) \vee (s_{1k}, s_{2k}) \leq (a_1, a_2) \vee ((b_1, b_2) \wedge (c_1, c_2)) = (a_1, a_2)$. Otherwise $(a_1, a_2) \vee (r_{1k}, r_{2k}) = (a_1, a_2) \vee (s_{1k}, s_{2k}) = (a_1, a_2)$ by the join-semidistributivity of $L_i (i = 1, 2)$. Hence $(p, q) \leq (a_1, a_2)$ in $L_1 \square L_2$, showing that $(b_1, b_2) \leq (a_1, a_2)$.

We recall that, in any lattice L, the following conditions are equivalent:

(i) For any elements $x_1, x_2, \cdots, x_n \in L$, if $x_i \wedge x_j \neq 0$ for all $i \neq j$, then $x_1 \wedge x_2 \wedge \cdots \wedge x_n \neq 0$ in L.

(ii) For any elements x, y, z in L, if each of $x \wedge y$, $y \wedge z$ and $z \wedge x$ is nonzero elements, then $x \wedge y \wedge z$ is nonzero.

A lattice L is said to satisfy the *Chinese Remainder Theorem*(C R T) if and only if the condition (i) (and hence (ii)) holds in it.

In general, the C R T need not be preserved by taking direct products. For example, M_n and **2** satisfy the C R T, but $M_n \times \mathbf{2}$ does not, where M_n is the lattice of length 2 with n atoms. But this property is preserved by taking rectangular products, as the definition of rectangular product that $(a, b) = (0, 0)$ in $L_1 \square L_2$ if and only if $a = 0$ in L_1 or $b = 0$ in L_2.

THEOREM 5. *The rectangular product of lattices L_1 and L_2 satisfies the C R T if and only if both L_1 and L_2 do.*

A lattice L satisfies the *strict Chinese Remainder Theorem* (strict C R T) if L is atomic and for any atoms p, q, r in L, $M(p,q,r) = (p \vee q) \wedge (q \vee r) \wedge (r \vee p)$ is an atom in L. Note that the strict C R T implies the C R T, but the converse does not hold (see M_n).

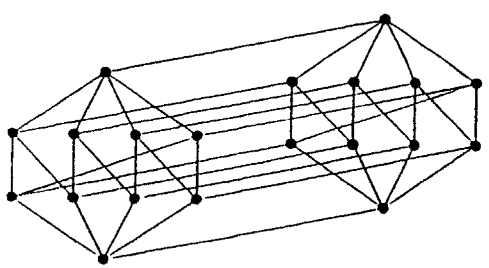

Figure 2.

In general, the strict C R T need not be preserved by taking direct products. For example, B_n and $\mathbf{2}$ satisfy the strict C R T, but $B_n \times \mathbf{2}$ does not(see Figure 2 for $n = 4$), where $A_n : a_1 < b_1 > a_2 < b_2 > \cdots < b_{n-1} > a_n < b_n > a_1$ and $B_n = A_n \cup \{0, 1\}$. But this property is preserved by rectangular products, as the following theorem shows.

THEOREM 6. *The rectangular product of lattices L_1 and L_2 satisfies the strict C R T if and only if both L_1 and L_2 do.*

Proof. Suppose that L_1 and L_2 satisfy the strict C R T. Note that L_1 and L_2 are atomic if and only if $L_1 \,\square\, L_2$ is an atomic by Theorem A. We know that $(p_1, p_2), (q_1, q_2)$ and (r_1, r_2) are atoms in $L_1 \,\square\, L_2$ if and only if p_i, q_i, r_i are atoms in L_i for i=1,2. Since L_1 and L_2 satisfy the strict C R T, $M(p_1, q_1, r_1)$ is an atom in L_1 and $M(p_2, q_2, r_2)$ is an atom in L_2. Recall that the definition of rectangular product that $L_1 \,\square\, L_2$ is a join sublattice of the direct product $L_1 \times L_2$ and that any nonzero meet agrees in the rectangular product and the direct product. Now we have, in $L_1 \square L_2$,

$$\begin{aligned}
&M((p_1,p_2),(q_1,q_2),(r_1,r_2)) \\
&= ((p_1,p_2) \vee (q_1,q_2)) \wedge ((q_1,q_2) \vee (r_1,r_2)) \wedge ((r_1,r_2) \vee (p_1,p_2)) \\
&= (p_1 \vee q_1, p_2 \vee q_2) \wedge (q_1 \vee r_1, q_2 \vee r_2) \wedge (r_1 \vee p_1, r_2 \vee p_2).
\end{aligned}$$

Since $M(p_i,q_i,r_i) = (p_i \vee q_i) \wedge (q_i \vee r_i) \wedge (r_i \vee p_i)$ is an atom in L_i for $i = 1,2$. Thus $M((p_1,p_2),(q_1,q_2),(r_1,r_2)) = (M(p_1,q_1,r_1), M(p_2,q_2,r_2))$ is an atom in $L_1 \square L_2$. Hence $L_1 \square L_2$ satisfies the strict C R T.

Conversely, suppose that $L_1 \square L_2$ satisfies the strict C R T. Take any atoms p_1,q_1,r_1 in L_1. Thus $(p_1,s),(q_1,s)$ and (r_1,s) are atoms in $L_1 \square L_2$ for any atom s in L_2. Since $L_1 \square L_2$ satisfies the strict C R T, we have

$$\begin{aligned}
&M((p_1,s),(q_1,s),(r_1,s)) \\
&= ((p_1,s) \vee (q_1,s)) \wedge ((q_1,s) \vee (r_1,s)) \wedge ((r_1,s) \vee (p_1,s)) \\
&= (p_1 \vee q_1, s) \wedge (q_1 \vee r_1, s) \wedge (r_1 \vee p_1, s),
\end{aligned}$$

which is an atom in $L_1 \square L_2$. Thus

$$\begin{aligned}
&M((p_1,s),(q_1,s),(r_1,s)) \\
&= (p_1 \vee q_1, s) \wedge (q_1 \vee r_1, s) \wedge (r_1 \vee p_1, s) \\
&= ((p_1 \vee q_1) \wedge (q_1 \vee r_1) \wedge (r_1 \vee p_1), s) \\
&= (M(p_1,q_1,r_1), s),
\end{aligned}$$

which is an atom in $L_1 \square L_2$ and so $M(p_1,q_1,r_1)$ is an atom in L_1. Hence L_1 satisfies the strict C R T. Similarly, L_2 satisfies the strict C R T.

An algebraic lattice is said to be *median* if it is biatomic and satisfies the strict C R T.

THEOREM 7. *The rectangular product of lattices L_1 and L_2 is median if and only if both L_1 and L_2 are.*

Proof. It follows from Theorems C, 2 and 6

A lattice satisfies the *switching condition* if L is atomic and for any atoms p,q,r,s with $p,q \leq r \vee s$ and $p \leq r \vee q$ implies that $q \leq s \vee p$.

Since all joins, and nonzero meets agree in both $L_1 \square L_2$ and $L_1 \times L_2$, the following results are obtained immediately.

THEOREM 8. *The rectangular product of lattices L_1 and L_2 satisfies the switching condition if and only if so do L_1 and L_2.*

THEOREM 9. *The rectangular product of lattices L_1 and L_2 is complemented if and only if both L_1 and L_2 are.*

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