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FUZZY COMMUTATIVE IDEALS IN BCI-ALGEBRAS

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In [6], the second author of this paper and X. L. Xin introduced the notion of commutative BCI-algebras and discussed the structure of it. Also the second author [4] introduced the concept of commutative ideals in BCK-algebras, and the results in [4] are generalized to BCI-algebra by him([5]). In [3] the first author of this paper introduced the notion of closed fuzzy ideals of BCI-algebras and studied their properties.

The aim of this paper is to introduce the concept of fuzzy commutative ideals of BCI-algebras, and is to study their properties.

By a BCI-algebra we mean a nonempty set X with a binary operation $*$ and a constant 0 satisfying the axioms:

BCI-1 $((x*y)*(x*z))*(z*y)=0$,

BCI-2 $(x*(x*y))*y=0$,

BCI-3 $x*x=0$,

BCI-4 $x*y=0$ and $y*x=0$ imply $x=y$,

BCI-5 $x*0=0$ implies that $x=0$,

for all $x,y,z\in X$. In a BCI-algebra X, we can define a partial ordering $\leq$ by putting $x\leq y$ if and only if $x*y=0$.

In any BCI-algebra X, the following hold:

(1) $(x*y)*z=(x*z)*y$,

(2) $x*(x*(x*y))=x*y$,

(3) $x*0=x$.

In this paper, unless otherwise specified, X will always mean a BCI-algebra.

A nonempty subset I of X is called an ideal if it satisfies

(I1) $0\in I$,

(I2) $x*y\in I$ and $y\in I$ imply $x\in I$.

We recall that a fuzzy set in X is a function μ from X into [0, 1].

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Definition 1 ([7]). A fuzzy set μ in X is called a fuzzy ideal if it satisfies

(F1) $\mu(0) \geq \mu(x)$ for all $x \in X$,

(F2) $\mu(x) \geq \min\{\mu(x * y), \mu(y)\}$ for all $x, y \in X$.

Definition 2 ([3]). A fuzzy ideal μ of X is said to be closed if $\mu(0*x) \geq \mu(x)$ for all $x \in X$.

Now, for any fuzzy set μ in X, consider the following condition:

(4) $x * y \leq z$ implies $\mu(x) \geq \min\{\mu(y), \mu(z)\}$

for all $x, y, z \in X$.

THEOREM 1. *Let μ be a fuzzy set in X. Then*

(a) *if μ is a fuzzy ideal of X, then μ satisfies the condition* (4).

(b) *if μ satisfies the conditions* (F1) *and* (4), *then μ is a fuzzy ideal of X.*

Proof. (a). Let μ be a fuzzy ideal of X and let $x, y, z \in X$ be such that $x * y \leq z$. Then

$$\begin{aligned}\mu(x * y) &\geq \min\{\mu((x * y) * z), \mu(z)\}\\ &= \min\{\mu(0), \mu(z)\}\\ &= \mu(z).\end{aligned}$$

It follows that

$$\mu(x) \geq \min\{\mu(x * y), \mu(y)\} \geq \min\{\mu(y), \mu(z)\}.$$

Thus μ satisfies the condition (4).

(b). Assume that μ satisfies the conditions (F1) and (4). Combining BCI-2 and (4), we have

$$\mu(x) \geq \min\{\mu(x * y), \mu(y)\}$$

for all $x, y \in X$. This is the condition (F2) and the proof is complete.

Definition 3 ([5]). A nonempty subset I of X is called a commutative ideal if it satisfies (I1) and

(C1) $(x*y)*z \in I$ and $z \in I$ imply $x*((y*(y*x))*(0*(0*(x*y)))) \in I$

for all x, y, z in X.

Definition 4. A fuzzy set μ in X is called a fuzzy commutative ideal if it satisfies (F1) and

(FC1) $\mu(x*((y*(y*x))*(0*(0*(x*y))))) \geq \min\{\mu((x*y)*z), \mu(z)\}$ for all $x, y, z \in X$.

Example. Let $X = \{0, a, 1, 2, 3\}$ in which $*$ is defined by

*	0	a	1	2	3
0	0	0	3	2	1
a	a	0	3	2	1
1	1	1	0	3	2
2	2	2	1	0	3
3	3	3	2	1	0

Then $(X; *, 0)$ is a BCI-algebra([1]). Let $t_0, t_1, t_2 \in [0, 1]$ be such that $t_0 > t_1 > t_2$. Define a function $\mu : X \to [0, 1]$ by $\mu(0) = t_0$, $\mu(a) = t_1$ and $\mu(1) = \mu(2) = \mu(3) = t_2$. By routine calculations, we know that μ is a fuzzy commutative ideal of X.

The following theorem is obvious, and omit the proof.

THEOREM 2. *Let I be any nonempty subset of X. Then the following conditions are equivalent:*

(a) *I is a commutative ideal of X.*

(b) *The characteristic function χ_I of I is a fuzzy commutative ideal of X.*

THEOREM 3. *Any fuzzy commutative ideal must be a fuzzy ideal.*

Proof. Let μ be a fuzzy commutative ideal of X and let $x, z \in X$. Then

$$\begin{aligned} &\min\{\mu(x*z), \mu(z)\} \\ &= \min\{\mu((x*0)*z), \mu(z)\} \\ &\leq \mu(x*((0*(0*x))*(0*(0*(x*0))))) \\ &= \mu(x*0) \\ &= \mu(x). \end{aligned}$$

This completes the proof.

Remark. A fuzzy ideal may not be fuzzy commutative. For instance, let $X = \{0, 1, 2, 3, 4\}$ in which the operation $*$ is defined by

*	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	0
2	2	2	0	0	0
3	3	3	3	0	0
4	4	4	4	3	0

Then $(X; *, 0)$ is a BCI-algebra([5]). Let $t_0, t_1, t_2 \in [0, 1]$ be such that $t_0 > t_1 > t_2$. Define $\mu : X \to [0, 1]$ by $\mu(0) = t_0$, $\mu(1) = t_1$ and $\mu(2) = \mu(3) = \mu(4) = t_2$. Routine calculations give that μ is a fuzzy ideal of X. But μ is not a fuzzy commutative ideal of X, because

$$\mu(2 * ((3 * (3 * 2)) * (0 * (0 * (2 * 3))))) \not\geq \min\{\mu((2 * 3) * 0), \mu(0)\}.$$

THEOREM 4. *A fuzzy ideal μ of X is fuzzy commutative if and only if it satisfies*

(FC2) $\mu(x * ((y * (y * x)) * (0 * (0 * (x * y))))) \geq \mu(x * y)$

for all $x, y \in X$.

Proof. Assume that μ is a fuzzy commutative ideal of X. If we take $z = 0$ in (FC1), then we get (FC2).

Conversely suppose that μ satisfies (FC2). Since μ is a fuzzy ideal, therefore

$$\mu(x * y) \geq \min\{\mu((x * y) * z), \mu(z)\}$$

for all $x, y, z \in X$. From (FC2), we obtain (FC1). This completes the proof.

THEOREM 5. *Let μ be a closed fuzzy ideal of X. Then μ is fuzzy commutative if and only if it satisfies*

(FC3) $\mu(x * (y * (y * x))) \geq \mu(x * y)$ *for all $x, y \in X$.*

Proof. Suppose that μ is a fuzzy commutative ideal of X. Let $x, y, z \in$

X. Then

$$\begin{aligned}&(x*(y*(y*x)))*(x*((y*(y*x))*(0*(0*(x*y)))))\\ &\leq ((y*(y*x))*(0*(0*(x*y))))*(y*(y*x))\\ &=((y*(y*x))*(y*(y*x)))*(0*(0*(x*y)))\\ &=0*(0*(0*(x*y)))\\ &=0*(x*y).\end{aligned}$$

It follows from Theorem 1(a) and (FC2) that

$$\begin{aligned}&\mu(x*(y*(y*x)))\\ &\geq \min\{\mu(x*((y*(y*x))*(0*(0*(x*y))))),\mu(0*(x*y))\}\\ &\geq \min\{\mu(x*y),\mu(0*(x*y))\}\\ &=\mu(x*y).\end{aligned}$$

Hence μ satisfies (FC3).

Conversely assume that μ satisfies the condition (FC3) and let $x, y \in X$. Then

$$\begin{aligned}&(x*((y*(y*x))*(0*(0*(x*y)))))*(x*(y*(y*x)))\\ &\leq (y*(y*x))*((y*(y*x))*(0*(0*(x*y))))\\ &\leq 0*(0*(x*y)).\end{aligned}$$

By Theorem 1(a) and (FC3) we have

$$\begin{aligned}&\mu(x*((y*(y*x))*(0*(0*(x*y)))))\\ &\geq \min\{\mu(x*(y*(y*x))),\mu(0*(0*(x*y)))\}\\ &\geq \min\{\mu(x*y),\mu(0*(0*(x*y)))\}\\ &=\mu(x*y),\end{aligned}$$

which proves that μ satisfies the condition (FC2). Hence μ is fuzzy commutative, and the proof is complete.

Definition 5 ([6]). A BCI-algebra X is called commutative if, for all x, y in X,

(5) $x \leq y$ implies $x = y*(y*x)$.

PROPOSITION 1 ([6]). *A BCI-algebra X is commutative if and only if it satisfies*

(6) $x * (x * y) = y * (y * (x * (x * y)))$ *for all* $x, y \in X$.

THEOREM 6. *Let X be a commutative BCI-algebra. Then every closed fuzzy ideal is fuzzy commutative.*

Proof. Assume that X is a commutative BCI-algebra and let μ be a closed fuzzy ideal of X. Let $x, y \in X$. Then, by means of Proposition 1, we have

$$\begin{aligned}
&(x * (y * (y * x))) * (x * y) \\
&= (x * (x * y)) * (y * (y * x)) \\
&= (y * (y * (x * (x * y)))) * (y * (y * x)) \\
&= (y * (y * (y * x))) * (y * (x * (x * y))) \\
&= (y * x) * (y * (x * (x * y))) \\
&\leq (x * (x * y)) * x \\
&= 0 * (x * y).
\end{aligned}$$

Using Theorem 1(a); we get

$$\mu(x * (y * (y * x))) \geq \min\{\mu(x * y), \mu(0 * (x * y))\} = \mu(x * y).$$

Hence, by Theorem 5 we conclude that μ is a fuzzy commutative ideal of X.

References

1. S. A. Bhatti, M. A. Chaudhry and B. Ahmad, *On classification of BCI-algebras*, Math. Japon. **34** (1989), 865–876.
2. Y. B. Jun, *Characterization of fuzzy ideals by their level ideals in BCK(BCI)-algebras*, Math. Japon. **38** (1993), 67–71.
3. Y. B. Jun, *Closed fuzzy ideals in BCI-algebras*, Math. Japon. **38** (1993), 199–202.
4. J. Meng, *Commutative ideals in BCK-algebras*, Pure appl. Math. (in China) **9** (1991), 49–53.
5. J. Meng, *An ideal characterization of commutative BCI-algebras*, Pusan Kyongnam Math. J. **9** (1993), 1–6.
6. J. Meng and X. L. Xin, *Commutative BCI-algebras*, Math. Japon. **37** (1992), 569–572.

7. O. G. Xi, *Fuzzy BCK-algebra*, Math. Japon. **36** (1991), 935–942.

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