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GENERALIZED KO'S COMPLEXES

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1. Introduction

In [3] and [5], the Schur functors and coSchur functors were introduced as the functorial generalization of the Schur and Weyl modules, respectively, in the polynomial representation theory of the general linear group. They are defined over any commutative ring and they are parametrized by partitions. Special cases of Schur functors arise as cycles in a generic Koszul complexes [4]. K.Akin and D.A. Buchsbaum used the Schur functors to generalize the Koszul complex [1].

It turns out that they have found a wide range of applications, ranging from the theory of free resolutions to the symmetric function theory. Recently, K.Akin and D.A. Buchsbaum [2] realized the Jacobi-Trudi identity for Schur functions as a resolution in the category of polynomial representations of the general linear group. T. Józefiak and J.Weyman [6] used the Koszul complex to realize a formula of D.E. Littlewood as a resolution of Schur modules.

In this article we will describe new classes of finite free resolutions related to any skew partitions. As special cases of these finite free resolutions we obtained the generalized Koszul complex constructed in [1].

Throughout this paper R will denote a commutative ring with identity and F a finitely generated free R-module.

2. Fundamental short exact sequences

On the characteristic-free representation theory of the general linear group,we use the definition and the notation of [3] freely. But we shall review some facts which will be used later. For the details, see [2] and [3].

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Let F be a free R-module of rank n. Then we have the complex

$$\cdots \longrightarrow S_{p-1}F \otimes \Lambda^{q+1}F \xrightarrow{\delta} S_pF \otimes \Lambda^q F \xrightarrow{\delta} S_{p+1}F \otimes \Lambda^{q-1}F \longrightarrow \cdots$$

which may be regarded as one of the strands of the Koszul complex associated to the ideal $(x_1, \ldots, x_n)$ in the polynomial ring $R[x_1, \ldots, x_n]$. Notice that the boundary map $\delta : S_pF \otimes \Lambda^q F \longrightarrow S_{p+1}F \otimes \Lambda^{q-1}F$ is defined as the composite

$$S_pF \otimes \Lambda^q F \xrightarrow{1\otimes\Delta} S_pF \otimes F \otimes \Lambda^{q-1}F \xrightarrow{m\otimes 1} S_{p+1}F \otimes \Lambda^{q-1}F$$

where Δ is the appropriate component of the comultiplication of the exterior algebra ΛF and m is the multiplication on the symmetric algebra SF. The image of the map $\delta : S_pF \otimes \Lambda^q F \longrightarrow S_{p+1}F \otimes \Lambda^{q-1}F$ is denoted by $L_{(q,\underbrace{1,\cdots,1}_{p})}F$ in [3], and is the Schur module associated with the partition $(q, \underbrace{1, \cdots, 1}_{p})$, or simply $(q, 1^p)$.

Thus we may regard the Koszul complex as a resolution of $L_{(q,1^p)}F$:

$$\begin{aligned} 0 \longrightarrow S_{p+q-n}F \otimes \Lambda^n F \longrightarrow \cdots \longrightarrow S_{p-1}F \otimes \Lambda^{q+1}F \\ \longrightarrow S_pF \otimes \Lambda^q F \longrightarrow L_{(q,1^p)}F \longrightarrow 0. \end{aligned}$$

Now let $\alpha = \lambda/\mu$ be any skew partition and $S_pF \otimes L_\alpha F \longrightarrow L_{(\lambda,1^p)/\mu}F$ the natural surjection (to be defined below). Then we will show in the next section that the following complex is exact :

(1) $\cdots \to S_{p-2}F \otimes X_2 \to S_{p-1}F \otimes X_1 \to S_pF \otimes L_\alpha F \to L_{(\lambda,1^p)/\mu}F \to 0.$

where the X_i are Schur modules depending on α in such a way that, when $L_\alpha F = \Lambda^q F$, we have $X_i = \Lambda^{q+i}F$.

Observing that $\Lambda^q F = L_{(q)}F$, $S_pF = L_{(1^p)}F$, and $L_\lambda F = L_{\lambda/(0)}F$ (for any partition λ), the complex (1) is the obvious generalization of the Koszul complex.

To facilitate matters, we introduce some notations. If $\lambda = (\lambda_1, \ldots, \lambda_r)$ is a partition and l is a nonnegative integer, we shall denote by $\lambda + l$ the partition $(\lambda_1 + l, \ldots, \lambda_r + l)$. It should be noticed that $\lambda + l$ is denoted by the skew partition $(\lambda_1 + l, \ldots, \lambda_r + l)/(l^{r-1})$ in [1]. We also denote by (λ, k) the partition $(\lambda_1, \lambda_2, \ldots, \lambda_r, k)$ and by $(\lambda, 1^k)$ the partition $(\lambda_1, \ldots, \lambda_r, \underbrace{1, \cdots, 1}_{k})$ for a nonnegative integer k.

LEMMA 2.1. *Let $\lambda/\mu = (\lambda_1, \ldots, \lambda_r)/(\mu_1, \ldots, \mu_r)$ be any skew partition with $\lambda_1 = q$. Then the Schur map*

$$d_{(\lambda,1^p)/\mu}F : \Lambda^{\lambda_1-\mu_1}F \otimes \cdots \otimes \Lambda^{\lambda_r-\mu_r}F \otimes \underbrace{F \otimes \cdots \otimes F}_{p} \longrightarrow S_{\tilde{\lambda}_1-\tilde{\mu}_1+p}F \otimes S_{\tilde{\lambda}_2-\tilde{\mu}_2}F \otimes \cdots \otimes S_{\tilde{\lambda}_q-\tilde{\mu}_q}F$$

is the composite

$$\begin{aligned}
&\Lambda^{\lambda_1-\mu_1}F \otimes \cdots \otimes \Lambda^{\lambda_r-\mu_r}F \otimes \underbrace{F \otimes \cdots \otimes F}_{p} \\
&\xrightarrow{1\otimes\cdots\otimes 1\otimes m} \Lambda^{\lambda_1-\mu_1}F \otimes \cdots \otimes \Lambda^{\lambda_r-\mu_r}F \otimes S_pF \\
&\xrightarrow{d_{\lambda/\mu}F\otimes 1} S_{\tilde{\lambda}_1-\tilde{\mu}_1}F \otimes \cdots \otimes S_{\tilde{\lambda}_q-\tilde{\mu}_q}F \otimes S_pF \\
&\cong S_{\tilde{\lambda}_1-\tilde{\mu}_1}F \otimes S_pF \otimes S_{\tilde{\lambda}_2-\tilde{\mu}_2}F \otimes \cdots \otimes S_{\tilde{\lambda}_q-\tilde{\mu}_q}F \\
&\xrightarrow{m\otimes 1\otimes\cdots 1} S_{\tilde{\lambda}_1-\tilde{\mu}_1+p}F \otimes S_{\tilde{\lambda}_2-\tilde{\mu}_2}F \otimes \cdots \otimes S_{\tilde{\lambda}_q-\tilde{\mu}_q}F
\end{aligned}$$

Proof. It follows immediately from the associativity of multiplication m in the Hopf algebra SF.

Now we will define the natural surjection of $S_pF \otimes L_{\lambda/\mu}F \to L_{(\lambda,1^p)/\mu}F$. Since $1 \otimes \cdots \otimes 1 \otimes m$ is surjective, the image of $d_{\lambda/\mu}F \otimes 1$ clearly gets mapped surjectively onto the image of $d_{(\lambda,1^p)/\mu}F$ and it is this surjection of $S_pF \otimes L_{\lambda/\mu}F$ onto $L_{(\lambda,1^p)/\mu}F$ which generalizes the surjection of $S_pF \otimes \Lambda^qF$ onto $L_{(q,1^p)}F$.

In order to construct the generalized Koszul complex we need the fundamental short exact sequences of Schur modules.

DEFINITION 2.2. Let α_1, α_2, and l be positive integers with $\alpha_2 \geq l$. We define a GL(F)-morphism

$$\square_l(F) : \Lambda^{\alpha_1+l}F \otimes \Lambda^{\alpha_2-l}F \longrightarrow \Lambda^{\alpha_1}F \otimes \Lambda^{\alpha_2}F$$

as the composite of GL(F)-morphisms

$$\Lambda^{\alpha_1+l}F \otimes \Lambda^{\alpha_2-l}F \xrightarrow{\Delta\otimes 1} \Lambda^{\alpha_1}F \otimes \Lambda^{l}F \otimes \Lambda^{\alpha_2-l}F \xrightarrow{1\otimes m} \Lambda^{\alpha_1}F \otimes \Lambda^{\alpha_2}F$$

where Δ is the indicated diagonal map and m is the multiplication map. Similarly when $\alpha = (\alpha_1, \ldots, \alpha_r)$ is a sequence of positive integers, we define a GL(F)-morphsim as

$$\sum_{i=1}^{r-1}\sum_{l=1}^{\alpha_{i+1}} \Lambda^{\alpha_1}F\otimes\cdots\otimes\Lambda^{\alpha_{i-1}}F\otimes\Lambda^{\alpha_i+l}F\otimes\Lambda^{\alpha_{i+1}-l}F\otimes\Lambda^{\alpha_{i+2}}F\otimes\cdots\otimes\Lambda^{\alpha_r}F$$

$$\Big\downarrow \quad \sum_{i=1}^{r-1}\sum_{l=1}^{\alpha_{i+1}} 1\otimes\cdots\otimes\Box_l(F)\otimes 1\otimes\cdots\otimes 1$$

$$\Lambda\alpha F = \Lambda^{\alpha_1}F\otimes\cdots\otimes\Lambda^{\alpha_r}F$$

and denote it by $\Box_\alpha(F)$ or simply $\Box_\alpha$.

PROPOSITION 2.3 ([3]). *For any skew partition* $\alpha = (\lambda_1, \ldots, \lambda_r)/(\mu_1, \ldots, \mu_r)$, *the following sequence of GL(F)-morphisms is exact :*

$$0 \longrightarrow Im(\Box_\alpha(F)) \longrightarrow \Lambda_\alpha F \overset{d_\alpha(F)}{\longrightarrow} L_\alpha(F) \longrightarrow 0.$$

Now let $\alpha = (\lambda_1, \ldots, \lambda_r)/(\mu_1, \ldots, \mu_r)$ be any skew partition with $\lambda_r \neq 0$ and $r > 2$. Then we will show that there is a fundamental short exact sequence of Schur modules

$$0 \longrightarrow L_\gamma F \longrightarrow L_\beta F \longrightarrow L_\alpha F \longrightarrow 0$$

where $\beta = (\lambda_1, \ldots, \lambda_{r-1}, \lambda_r - 1)/(\mu_1, \ldots, \mu_{r-1}, \mu_r - 1)$ and $\gamma = (\lambda_1, \ldots, \lambda_{r-2}, \lambda_{r-1}, \lambda_r - 1)/(\mu_1, \ldots, \mu_{r-2}, \mu_r - 1, \mu_{r-1})$.

Notice that any skew partition can be represented by a relative sequence λ/μ of partitions λ and μ where $\mu_i \geq 1$ for all i so that we need not concern ourselves with negative entries. Observing that $\beta_i = \lambda_i - \mu_i = \alpha_i$ for all $i (i = 1, \ldots, r)$, we have the equalities $\Lambda_\beta F = \Lambda^{\lambda_1-\mu_1}F\otimes\cdots\otimes\Lambda^{\lambda_r-\mu_r}F = \Lambda_\alpha F$. Then the identity map of $\Lambda_\beta F \overset{=}{\longrightarrow} \Lambda_\alpha F$ on the generators induces a natural surjection $v : L_\beta F \longrightarrow L_\alpha F$. This can be seen by examining the relations for $L_\beta F$ and $L_\alpha F$. By Proposition 2.3, the relations for $L_\alpha F$ are given by the double summation

$$(2) \quad \sum_{i=1}^{r-1}\Big[\sum_{l>\mu_i-\mu_{i+1}}^{\alpha_{i+1}} \Lambda^{\alpha_1}F\otimes\cdots\otimes\Lambda^{\alpha_i+l}F\otimes\Lambda^{\alpha_{i+1}-l}F\otimes\cdots\otimes\Lambda^{\alpha_r}F\Big]$$

and analogously for $L_\beta F$. It is clear that $L_\beta F$ has exactly the same relations for $i < r-1$, and when $i = r-1$ the inner summation is

$$\sum_{l>\mu_{r-1}-\mu_r+1}^{\beta_r} \Lambda^{\beta_1}F \otimes \cdots \otimes \Lambda^{\beta_{r-1}+l}F \otimes \Lambda^{\beta_r - l}F.$$

Consequently the only difference between the two sets of relations is the term $\Lambda^{\gamma_1}F \otimes \cdots \otimes \Lambda^{\gamma_r}F$ with indices $i = r-1$ and $l = \mu_{r-1} - \mu_r + 1$ in the summation (2). Since every relation for $L_\beta F$ is also a relation for $L_\alpha F$ the identity map $\Lambda_\beta F \longrightarrow \Lambda_\alpha F$ induces a natural surjection $v : L_\beta F \longrightarrow L_\alpha F$ and thus the kernel of $v : L_\beta F \longrightarrow L_\alpha F$ must be the image of the composite map

$$\Lambda_\gamma F \xrightarrow{\square} \Lambda_\beta F \xrightarrow{d_\beta(F)} L_\beta F \tag{3}$$

where $\square$ is the indicated restriction of $\square_\beta$.

Now we claim that the image of $d_\beta F \circ \square$ in (3) is $L_\gamma F$. To see this, we first observe that $S_{\tilde{\gamma}}F = S_{\tilde{\beta}}F$. Second, the diagram

$$\begin{array}{ccc} \Lambda_\gamma F & \xrightarrow{\square} & \Lambda_\beta F \\ {\scriptstyle d_\gamma(F)}\downarrow & & \downarrow{\scriptstyle d_\beta(F)} \\ S_{\tilde{\gamma}}F & \xrightarrow{=} & S_\beta F \end{array} \tag{4}$$

is commutative. The commutativity of (4) is a straightforward computation or can be deduced from the Lemma II.2.5, 6, and 7 of [3]. Hence we have the following result.

THEOREM 2.4. *Let* $\alpha = (\lambda_1, \ldots, \lambda_r)/(\mu_1, \ldots, \mu_r)$ *be a skew partition with* $\lambda_r \neq 0$. *Then*
(a) *If* $r = 2$, *then there exists a short exact sequence of Schur modules*

$$0 \longrightarrow L_{(\lambda_1-\mu_2+1,\lambda_2-\mu_1-1)}F \xrightarrow{u} L_{(\lambda_1+1,\lambda_2)/(\mu_1+1,\mu_2)}F \xrightarrow{v} L_\alpha F \longrightarrow 0.$$

(b) *If* $r > 2$, *then there exists a short exact sequence of Schur modules*

$$0 \longrightarrow L_\gamma F \xrightarrow{u} L_\beta F \xrightarrow{v} L_\alpha F \longrightarrow 0$$

where $\beta = (\lambda_1, \ldots, \lambda_{r-1}, \lambda_r - 1)/(\mu_1, \ldots, \mu_{r-1}, \mu_r - 1)$ *and* $\gamma = (\lambda_1, \ldots, \lambda_{r-2}, \lambda_{r-1}, \lambda_r - 1)/(\mu_1, \ldots, \mu_{r-2}, \mu_r - 1, \mu_{r-1})$.

Proof. The proof of the case $r = 2$ also follows by the exactly same reasoning as we have discussed for the case $r > 2$.

3. Generalized Koszul complexes

In this section we describe new classes of finite free resolutions of certain Schur modules which are related to the skew partitions λ/μ.

Now let k and l be nonnegative integers and let $\lambda/\mu = (\lambda_1, \ldots, \lambda_r)/(\mu_1, \ldots, \mu_r)$ be any skew partition of length r.
Then the Koszul map

$$\delta : S_{k-l}F \otimes \Lambda^{l+1}F \longrightarrow S_{k-l+1}F \otimes \Lambda^l F$$

induces the map

$$\begin{aligned} & S_{k-l}F \otimes \Lambda^{\lambda_1-\mu_1}F \otimes \cdots \otimes \Lambda^{\lambda_r-\mu_r}F \otimes \Lambda^{l+1}F \\ \cong\ & \Lambda^{\lambda_1-\mu_1}F \otimes \cdots \otimes \Lambda^{\lambda_r-\mu_r}F \otimes S_{k-l}F \otimes \Lambda^{l+1}F \\ \xrightarrow{1\otimes\delta}\ & \Lambda^{\lambda_1-\mu_1}F \otimes \cdots \otimes \Lambda^{\lambda_r-\mu_r}F \otimes S_{k-l+1}F \otimes \Lambda^l F \\ \cong\ & S_{k-l+1}F \otimes \Lambda^{\lambda_1-\mu_1}F \otimes \cdots \otimes \Lambda^{\lambda_r-\mu_r}F \otimes \Lambda^l F. \end{aligned}$$

Then we have a canonical map

$$\partial : S_{k-l}F \otimes L_{(\lambda,1)+l/\mu+l}F \longrightarrow S_{k-l+1}F \otimes L_{(\lambda,1)+l-1/\mu+l-1}F$$

which is induced, on the generator level, by the map $1 \otimes \delta$.

Indeed, if $x \otimes y_1 \otimes \cdots \otimes y_r \otimes y_{r+1}$ is any basis element of $S_{k-l}F \otimes \Lambda_{(\lambda,1)+l/\mu+l}F$ and $\Delta(y_{r+1}) = \sum_i y_{r+1\ i} \otimes y'_{r+1\ i} \in F \otimes \Lambda^l F$, then the map ∂ sends $x \otimes d_{(\lambda,1)+l/\mu+l}(F)(y_1 \otimes \cdots \otimes y_r \otimes y_{r+1})$ to $\sum_i x y_{r+1\ i} \otimes d_{(\lambda,1)+l-1/\mu+l-1}(F)(y_1 \otimes \cdots \otimes y_r \otimes y'_{r+1\ i})$.

We will now proceed to describe the generalized Koszul complex. To do this we need to reformulate the fundamental short exact sequence given in Theorem 2.4 in a form that we shall use. It is clear that for every nonnegative integer l and any skew partition $\lambda/\mu = (\lambda_1, \ldots, \lambda_r)/(\mu_1, \ldots, \mu_r)$ of length r. There is a natural isomorphism

$$\Lambda^l F \otimes L_{\lambda/\mu} F \cong L_{(\lambda,0)+l/\mu+l} F$$

because the Schur module $\Lambda^l F \otimes L_{\lambda/\mu} F$ is the image

$$d_{(\lambda,0)+l/\mu+l} F(\Lambda^l F \otimes \Lambda_{\lambda/\mu} F).$$

(Recall that $\Delta : \Lambda^l F \longrightarrow \underbrace{F \otimes \cdots \otimes F}_{l}$ is a split monomorphism) Thus we have the following result.

PROPOSITION 3.1. *Let l be any nonnegative integer and let $\lambda/\mu = (\lambda_1, \ldots, \lambda_r)/(\mu_1, \ldots, \mu_r)$ be any skew partition of length $r > 1$. Then there exists a short exact sequence of Schur modules*

$$0 \longrightarrow L_{\lambda+l/\mu'+l} F \xrightarrow{u} \Lambda^l F \otimes L_{\lambda/\mu} F \xrightarrow{v} L_{(\lambda,1)+l-1/\mu+l-1} F \longrightarrow 0$$

where $\mu' + l = (\mu_1 + l, \ldots, \mu_{r-1} + l, \mu_r)$.

Proof. The proof of the proposition follows easily from the above observation and Theorem 2.4.

We are now able to state and prove our main theorem.

THEOREM 3.2. *Let F be a finitely generated free module over a commutative ring R with unity, $\lambda/\mu = (\lambda_1, \ldots, \lambda_r)/(\mu_1, \ldots, \mu_r)$ any skew partition of length r, and p a nonnegative integer. Then the following sequence is exact :*

(5)

$$\begin{aligned} \cdots \longrightarrow S_{p-3} F \otimes L_{\lambda+3/\mu'+3} \xrightarrow{\partial} S_{p-2} F \otimes L_{\lambda+2/\mu'+2} F \\ \xrightarrow{\partial} S_{p-1} F \otimes L_{\lambda+1/\mu'+1} F \xrightarrow{\partial} S_p F \otimes L_{\lambda/\mu} F \xrightarrow{\partial} L_{(\lambda,1^p)/\mu} F \longrightarrow 0 \end{aligned}$$

Proof. First observe that when $r = 1$. $L_{\lambda+l/\mu+l} F = \Lambda^{\lambda_1 - \mu_1} F$, so that the sequence (5) reduces to the Koszul complex in that case. Second the case of $p = 0$ is trivial.

The proof proceeds by induction on p. The case $p = 1$ is the special case of Proposition 3.1 in which $l = 1$.
Assuming now that the theorem is true for p and the skew partition $(\lambda, 1)/\mu$, we consider the maps of complexes

$$\begin{array}{ccccccccc} \cdots \longrightarrow & S_{p-2}F\otimes\Lambda^3F\otimes L_{\lambda/\mu}F & \xrightarrow{\delta\otimes 1} & S_{p-1}F\otimes\Lambda^2F\otimes L_{\lambda/\mu}F & \xrightarrow{\delta\otimes 1} & S_pF\otimes F\otimes L_{\lambda/\mu}F & \xrightarrow{\delta\otimes 1} & S_{p+1}F\otimes L_{\lambda/\mu}F & \longrightarrow 0 \\ & \downarrow{\scriptstyle 1\otimes v} & & \downarrow{\scriptstyle 1\otimes v} & & \downarrow{\scriptstyle 1\otimes v} & & \downarrow{\scriptstyle \partial} & \\ \cdots \longrightarrow & S_{p-2}F\otimes L_{(\lambda,1)+2/\mu+2}F & \xrightarrow{\partial} & S_{p-1}F\otimes L_{(\lambda,1)+1/\mu+1}F & \xrightarrow{\partial} & S_pF\otimes L_{(\lambda,1)/\mu}F & \xrightarrow{\partial} & L_{(\lambda,1^{p+1})/\mu}F & \longrightarrow 0. \end{array} \tag{6}$$

Then it is easy to see that this is indeed a commutative diagram. The kernels of the maps $1 \otimes v$ are, by Proposition 3.1, the modules $S_{p-l}F \otimes L_{\lambda+l/\mu'+l}F$ for $l \geq 1$. The induction hypothesis on p, and the acyclicity of the Koszul complex, tell us that the two complexes above are exact. Then the simple argument of the long exact sequence in homology associated to (6) and the diagram-chasing show that the complex

$$\begin{aligned} \cdots \longrightarrow S_{p-1}F \otimes L_{\lambda+2/\mu'+2}F &\xrightarrow{\delta\otimes 1} S_pF \otimes L_{\lambda+1/\mu'+1}F \\ &\xrightarrow{(\delta\otimes 1)\circ(1\otimes u)} S_{p+1}F \otimes L_{\lambda/\mu}F \xrightarrow{\partial} L_{(\lambda,1^{p+1})/\mu}F \longrightarrow 0 \end{aligned}$$

is exact, and so the induction is complete.

If we consider the case of $\mu = \underbrace{(0, \cdots, 0)}_{r}$ in Theorem 3.2, then the above theorem implies the well known result.

COROLLARY 3.3 ([1]). *If F is a free R-module, λ is a partition of length r, and p is a nonnegative integer,then*

$$\begin{aligned} \cdots \longrightarrow S_{p-2}F \otimes L_{\lambda+2/(2^{r-1})}F &\longrightarrow S_{p-1}F \otimes L_{\lambda+1/(1^{r-1})}F \\ &\longrightarrow S_pF \otimes L_\lambda F \longrightarrow L_{(\lambda,1^p)}F \longrightarrow 0 \end{aligned}$$

is an exact sequence.

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