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E-UNITARY PROBLEM OF CERTAIN INVERSE MONOIDS

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1. Introduction

By a *monoid* we mean a semigroup with an identity element. Suppose that M is the monoid defined by a presentation $\mathrm{Mon}[X; R]$, where X is a set of generators and R is a set of relations. Thus X consists of symbols $x_1, x_2, \ldots$ and R consists of pairs (u_i, v_i), $(i \in I)$, where u_i and v_i are monoid words in X. Then M is (isomorphic to) the quotient X^*/τ, where X^* is the free monoid on X and τ is the smallest congruence relation on X^* containing R. We often write $M = \mathrm{Mon}[X; u_i = v_i,\ (i \in I)]$ to denote this monoid.

An *inverse monoid* is a monoid M with the property that, for each $a \in M$, there is a unique element denoted by a^{-1} in M such that

$$a = aa^{-1}a \text{ and } a^{-1} = a^{-1}aa^{-1}.$$

An inverse monoid can be alternatively defined as a universal algebra equipped with a binary operation $(x, y) \mapsto xy$, a unary operation $x \mapsto x^{-1}$ and a nullary operation selecting the identity element 1 of the monoid. From this point of view, inverse monoids form an equational class (or variety) of algebras of type $\langle 2, 1, 0\rangle$ subject to the following laws:

$$(xy)z = x(yz),\ x1 = x,\ 1x = x,\ xx^{-1}x = x,\ (x^{-1})^{-1} = x,$$

$$(xy)^{-1} = y^{-1}x^{-1}, \text{ and } (xx^{-1})(yy^{-1}) = (yy^{-1})(xx^{-1}).$$

Thus, as free algebras of this equational class, free inverse monoids exist. We denote the free monoid on X by $\mathrm{FIM}(X)$. In fact $\mathrm{FIM}(X)$ can be

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constructed as follow. Let X^{-1} be a set disjoint from X and in one-to-one correspondence with X by the map $x \mapsto x^{-1}$ $(x \in X)$, and ρ be the smallest congruence relation $(X \cup X^{-1})^*$ such that the above laws hold in $(X \cup X^{-1})^*/\rho$. Then $\mathrm{FIM}(X) \cong (X \cup X^{-1})^*/\rho$. The congruence relation ρ is called the *Vagner congruence.*

Let $R = \{(u_i, v_i) \mid i \in I\}$ be a set of relations on $(X \cup X^{-1})^*$. We define the inverse monoid presented by the set X of generators and the set R of relations, denoted by $M = \mathrm{Inv}[X; R]$, to be the inverse monoid $(X \cup X^{-1})^*/\theta$, where θ is the smallest congruence relation on $(X \cup X^{-1})^*$ generated by $\rho \cup R$. We sometimes write this presentation as $\mathrm{Inv}[X; u_i = v_i \ (i \in I)]$.

Let $\mathrm{FG}(X)$ denote the free group on X. Then the group defined by the presentation $\mathrm{Gp}\langle X; R\rangle$, is (isomorphic to) the quotient $\mathrm{FG}(X)/N$, where N is the smallest normal subgroup of $\mathrm{FG}(X)$ containing $\{uv^{-1} \mid (u, v) \in R\}$. A congruence relation π on an inverse monoid M is called the *minimal group congruence* if π is the smallest congruence relation such that M/π is a group; in this case, M/π is called the *maximum group homomorphic image* of M. The following lemma is easy.

LEMMA 1. *The group G defined by the presentation $Gp\langle X; R\rangle$ is the maximum group homomorphic image of the inverse monoid $M = Inv[X; R]$.*

Since idempotent elements of an inverse monoid are mapped to idempotent elements by homomorphisms, it is trivial that every idempotent element of an inverse monoid is mapped to the identity element by the maximum group homomorphism. However, not only idempotent elements are mapped to the identity by this homomorphism, but some other elements may also be mapped to the identity element. We call an inverse monoid *E-unitary* if only idempotent elements are mapped to the identity by the maximum group homomorphism. The problem we are interested in can be phrased as follow.

QUESTION. *When is an inverse monoid E-unitary?*

This problem has been known as E-unitary problem of inverse monoids and is far from being settled. Answering this question for inverse monoids given by a presentation is even more difficult. We state this problem as follow.

PROBLEM. *Find a condition on* R *so that the inverse monoid defined by the presentation* $\mathrm{Inv}[X;R]$ *is E-unitary.*

From now on, a 'word' will mean a word in $X \cup X^{-1}$. A word w in X is called *reduced* if w does not contain subwords of the form xx^{-1} or $x^{-1}x$ for any x in X. If $w = x_1^{\varepsilon_1}x_2^{\varepsilon_2}\cdots x_n^{\varepsilon_n}$ is a word with $\varepsilon_j = \pm 1$ for $j = 1,\ldots,n$, then a *cyclic permutation* of w is a word of the form

$$x_k^{\varepsilon_k}\cdots x_n^{\varepsilon_n}x_1^{\varepsilon_1}\cdots x_{k-1}^{\varepsilon_{k-1}}$$

for some $k = 1,\ldots,n$. A word is called *cyclically reduced* if every cyclic permutation of w is reduced.

In this paper, by a geometric methods using a kind of CW-complexes, called *diagrams*, we will prove the following theorem.

THEOREM. *Let* w *be a proper power, that is,* $w = u^n$ *for some word* u *and* $n > 1$. *Then the inverse monoid* $M = \mathrm{Inv}[X; w = 1]$ *is E-unitary.*

2. Diagrams

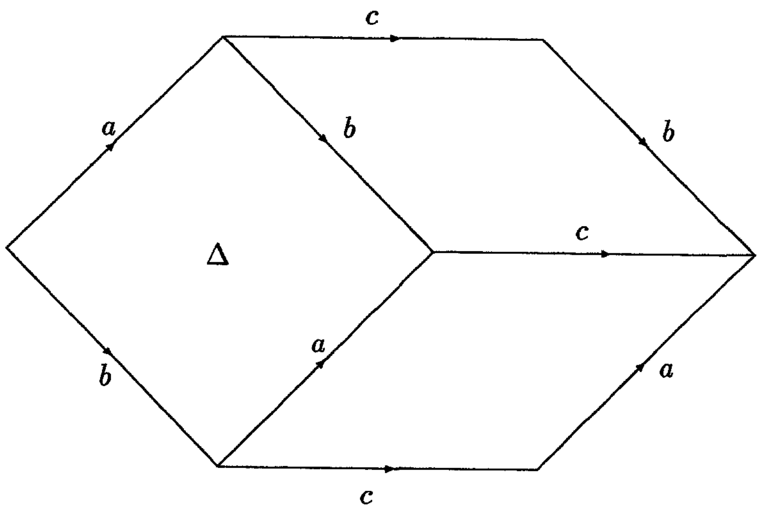
Let $\mathrm{Inv}[X;R]$ be the presentation of an inverse monoid M. By a *diagram* D over $M = \mathrm{Inv}[X;R]$, we mean a finite planar graph embedded in the plane $\mathbf{R}^2$ such that

(1) each edge of D is oriented and labelled by an element of X,
(2) if we read off the labels of a boundary of a connected region of $\mathbf{R}^2 \setminus D$, we get a word which is a cyclic permutation of a word uv^{-1} where $(u,v) \in R$ or $(v,u) \in R$.

It is a usual convention that when we read a boundary of a region counterclockwise, the label x of a clockwise oriented edge is read off as x^{-1}. For example, a diagram over the inverse monoid

$$M = \mathrm{Inv}[X; ab = ba, bc = ca, ca = ac]$$

with $X = \{a,b,c\}$ is given in the following figure. Here the boundary of the region Δ can be read off as $bab^{-1}a^{-1}$, $ab^{-1}a^{-1}b$, $b^{-1}a^{-1}ba$ or $a^{-1}bab^{-1}$, which are all cyclic permutations of $bab^{-1}a^{-1}$.



A *path* is a finite sequence of edges $e_1, e_2, \ldots, e_n$ such that the initial vertex of e_{i+1} is the terminal vertex of e_i for $i = 1, 2, \ldots, n-1$. A *loop* is a path which ends at the starting vertex of the path. By *label* of a path, we mean the word obtained by reading off labels of the edges in the path. A diagram is called *reduced* if it does not contain a loop such that reading of the edges along the loop yields a word of the form ww^{-1}.

The geometric method using diagrams is well developed for combinatorial group theory and proved to be very useful ([2], [3], [5]). This method is also adapted by many people for the study of word problem and embedding problem of semigroups ([1], [4], [11]). The following lemma is well-known ([5]).

LEMMA 2. *Let G be the group defined by the presentation $Gp\langle X; R\rangle$. A reduced word w represents the identity element of G if and only if there is a reduced diagram D over $Gp\langle X; R\rangle$ such that if we read off the boundary ∂D of D then we get a cyclic permutation of $w^{\pm 1}$.*

3. *E*-unitary inverse monoids

Let $\widehat{w}$ denote the reduced form of w, that is, $\widehat{w}$ is obtained from w by successively deleting any subword of the form xx^{-1} or $x^{-1}x$ for any x in X.

In this paper, we restrict our attention only to the case when R is consisting of only one relation of the form $w = 1$. Thus, suppose $M = \mathrm{Inv}[X; w = 1]$ with $w \in (X \cup X^{-1})^*$. We may regard w as an element of FIM(X). It is proved in [12] that there is an idempotent element e of FIM(X) such that $w = e\widehat{w}$ in FIM(X). Note that $e\widehat{w} = 1$ in M if and only if $e = 1$ and $\widehat{w} = 1$ in M, and so M is E-unitary if $\mathrm{Inv}[X; \widehat{w} = 1]$ is E-unitary. Thus, we restrict our attention to the case when w is reduced.

Let $M = \mathrm{Inv}[X; R]$ and $G = \mathrm{FG}(X)/N$ where N is the normal subgroup of FG(X) generated by $\{uv^{-1} \mid (u, v) \in R\}$. By Lemma 1, G is the maximal group homomorphic image of M. Let ϕ be the natural homomorphism from FIM(X) onto M. Then the following Lemma is trivial.

LEMMA 3. *M is E-unitary if and only if, for each word $u \in N$, $\phi(u)$ is an idempotent element of M.*

The following proposition is the best known-result so far for the E-unitary problem for inverse monoids, and is proved in [7] (also in [8]) by means of diagrams.

PROPOSITION 1. *Let w be a cyclically reduced word and $M = \mathrm{Inv}[X; w = 1]$. Then M is E-unitary if, for every reduced diagram D over M, there is at least one region Δ of D with an edge $e \in \partial D \cap \partial\Delta$ such that the cyclic permutation of $w^{\pm 1}$ obtained by reading off the labels of $\partial\Delta$ in the order starting from e represent 1 in M.*

4. Proof of Theorem

Suppose $w = u^n$ is a reduced word and $n > 1$. Then u is cyclically reduced, and so w is cyclically reduced; if u is not cyclically reduced then $w = uu \cdots u$ is not reduced. We may assume u is not a proper power, that is, u is not of the form v^k for any word v and integer $k > 1$. A *Gurevich subword* of w is a word of the form $v^{n-1}v'$, where v is a cyclic permutation of u or u^{-1}, and v' is an initial segment of v which contains every letter occurring in u. The following propositions proved in [10] (see also [3], [9]) is known as the spelling theorem.

PROPOSITION 2. *Let G be the group defined by $Gp\langle X; w\rangle$, where w is cyclically reduced. If t is a nonempty cyclically reduced word which* *defines 1 in G, then either t is a cyclic permutation of w or w^{-1}, or some cyclic permutation of t contains two disjoint subwords, each of which is a Gurevich subword of w.*

PROPOSITION 3. *Let G be the group defined by $Gp\langle X; w\rangle$, where w is cyclically reduced. If M is a reduced diagram over $Gp\langle X; w\rangle$ with at least two regions, then there are distinct regions Δ and Δ' of M and disjoint paths on $\partial M \cap \partial\Delta$ and $\partial M \cap \partial\Delta'$ such that the labels of each path yields a Gurevich subword of w.*

Now we are ready to prove our theorem.

Proof of Theorem. Let D be a reduced diagram over M. Let t be the word obtained by reading off the boundary of D. By Lemma 2, $t = 1$ in the group defined by the presentation $\mathrm{Gp}\langle X; w\rangle$. If D has only one region, then t is a conjugate of w and so there is an edge e in ∂D such that the word obtained by reading off ∂D stating from e is w, which is 1 in M. If D has at least two regions, then by Proposition 3 there is a region Δ such that $\partial D \cap \partial\Delta$ contains a path and the labels of edges in the path constitute a Gurevich subword of w. Since $w = u^n$ with $n > 1$, as a part of this path, there is a path on $\partial D \cap \partial\Delta$ such that the word obtained by reading off the path is u. Then the word obtained by reading off ∂D starting from the path is w, which is 1 in M. Thus, by Proposition 1, M is E-unitary.

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