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CONSTRUCTION OF HARMONIC MAPS BETWEEN PSEUDOSPHERES

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1. Introduction

Since the pioneer work [3] of Eells and Sampson, people have become to understand the harmonic maps between Riemannian manifolds. But in spite of the significance in the mathematical physics (ref. [4]), little is known about the harmonic maps between manifolds with indefinite metrics (ref. [2]). This lack of the existence result is due to the difficulty of the theory of hyperbolic partial differential equations.

We are interested in the existence of the harmonic maps between pseudo-Riemannian space forms, say, pseudospheres and pseudohyperbolic spaces. In view of the similarity between sphere and pseudosphere, we have tried to apply the method developed in the theory of the harmonic maps between spheres.

There are two importent classes of harmonic maps between spheres (ref. [1], [8]):

i) the harmonic maps obtained by Hopf construction;

ii) the harmonic maps homotopic to the join of two harmonic homogeneous polynomial maps (equivariant harmonic maps).

As for the Hopf construction, J. J. Konderak [6] obtained the harmonic homogeneous polynomial maps between pseudospheres and pseudohyperbolic spaces using the multiplication in some Clifford algebra. This harmonic map can easily be shown to be of constant energy density.

In fact, if a map $w: \mathbb{R}^{p,q} \to \mathbb{R}^{r,s}$ consists of harmonic k-homogeneous polynomials and $w(\Sigma(\varepsilon_1)) \subset \widetilde{\Sigma}(\varepsilon_2)$,then the map

$$\phi = w \,|_{\Sigma(\varepsilon_1)}: \Sigma(\varepsilon_1) \to \widetilde{\Sigma}(\varepsilon_2)$$

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is harmonic and $e(\phi) = \frac{1}{2}k(k+p+q-2)$.

Let (M, g) and (N, h) be pseudo-Riemannian manifolds and let $\phi: M \to N$ be a smooth map, then one can construct a bundle of 1-forms on M with values in the pull back bundle $\phi^{-1}TN$:

$$T^*M \otimes \phi^{-1}TN.$$

This bundle is equipped with the connection ∇ induced by the Levi-Civita connections on TM and TN, and the Hilbert-Schmidt product $< , >$. Considering $d\phi$ as a section of the bundle $T^*M \otimes \phi^{-1}TN \to M$, the energy density $e(\phi)$ is defined by $e(\phi) = (1/2) < d\phi, d\phi >$. The tension field of ϕ is defined by $\tau(\phi) = tr_g \ \nabla d\phi$, and ϕ is harmonic if and only if $\tau(\phi) = 0$.

Denote $\mathbb{R}^{m,n}$ $(m, n \geq 0)$ as the semi-Euclidean space with the metric

$$ds^2 = -\sum_{i=1}^{m} dx_i^2 + \sum_{j=m+1}^{m+n} dx_j^2.$$

For $\varepsilon \in \{-1, 1\}$ we define

$$\Sigma(\varepsilon) := \{ \ x \in \mathbb{R}^{p,q} \ | \ \langle \ x, x \ \rangle = \varepsilon \ \},$$

then $\Sigma(1) = S^{p,q-1}$, the pseudosphere, and $\Sigma(-1) = H^{p-1,q}$, the pseudohyperbolic space.

2. The structure of pseudosphere

There are several types of hyperbolic polar coordinates on pseudosphere and pseudohyperbolic space. For example, each point $z \in S^{p+1,q} \subset \mathbb{R}^{p+1,q+1}$ can be written as $z = (\sinh t \cdot u, \ \cosh t \cdot v)$, where $u \in S^p, v \in S^q$. So $S^{p+1,q}$ is isometric to

$$\left(S^p \times S^q \times [0, \infty), \ -\sinh^2 t \cdot g_1 + \cosh^2 t \cdot g_2 - dt^2\right),$$

where g_1 and g_2 are the standard metrics on S^p and S^q respectively (cf. [9]). This is a natural decomposition because the pseudosphere $S^{p+1,q}$ is isometric to the warped product

$$\left(\mathbb{R}^{p+1} \times S^q, \ -h + (1 + h(x,x))g\right),$$

where h is the Euclidean metric on $\mathbb{R}^{p+1}$, g is the standard metric on S^q and $x \in \mathbb{R}^{p+1}$.

Similarly we can easily see that each point $z \in H^{p,q+1} \subset \mathbb{R}^{p+1,q+1}$ is expressed as follows:

$$z = (\cosh t \cdot u,\ \sinh t \cdot v), \quad u \in S^p, v \in S^q$$

and so $H^{p,q+1}$ is isometric to

$$\left(S^p \times S^q \times [0,\infty),\ -\cosh^2 t \cdot g_1 + \sinh^2 t \cdot g_2 + dt^2\right),$$

where g_1, g_2 are the standard metrics on S^p, S^q, respectively.

Note that the locus $t = 0$ is a focal variety homeomorphic to S^p and the equivariant manifold $S^p \times S^q \times (0,\infty)$ is an open dense subset of $H^{p,q+1}$.

In the above coordinates, $S^{p+1,q}$ and $H^{p,q+1}$ are factored into spheres and a ray. In the following we are aim to factor into pseudospheres or pseudohyperbolic spaces, but things are not simple. In fact, pseudosphere consists of three parts, in each part there is an equivariant manifold of different type.

We introduce an isometry f from an equivariant manifold into a pseudosphere.

Let p, q, r, s be nonnegative integers and let

$$(\widetilde{M}, \widetilde{g}) = \left(S^{p+1,q} \times H^{r,s+1} \times [0,\infty),\ \cosh^2 t \cdot g_1 + \sinh^2 t \cdot g_2 - dt^2\right),$$

where g_1, g_2 are the standard metrics on $S^{p+1,q}$, $H^{r,s+1}$, respectively. Define a map

$$f : (\widetilde{M}, \widetilde{g}) \longrightarrow S^{p+r+2,q+s+1} \subset \mathbb{R}^{p+r+2,q+s+2}$$

by

$$f(x,y,t) = (\cosh t \cdot x,\ \sinh t \cdot y),$$

where $x \in S^{p+1,q}, y \in H^{r,s+1}, t \in [0,\infty)$. Then

$$\langle\ f(x,y,t), f(x,y,t)\ \rangle = \cosh^2 t\ \langle x,x\rangle + \sinh^2 t\ \langle y,y\rangle = 1.$$

Hence f is well defined.

LEMMA 2.1. *$f:(\widetilde{M},\widetilde{g}) \to S^{p+r+2,q+s+1}$ defined above is a one to one isometric embedding.*

Proof. It is easy to see that f is injective, and so we prove $\widetilde{g} = f^*g$ where g is the metric on $S^{p+r+2,q+s+1}$.

Let $(x_0, y_0, t) \in \widetilde{M}$ and let $\gamma: (-\delta, \delta) \to S^{p,q}$ $(\delta > 0)$ be a curve such that $\gamma(0) = x_0$. Define $\Gamma: (-\delta, \delta) \to \widetilde{M}$ by

$$\Gamma(s) = (\gamma(s), y_0, t_0), \quad s \in (-\delta, \delta),$$

then

$$|\, (f \circ \Gamma)'(0)\, |^2 = \cosh^2 t_0 \cdot \langle \gamma'(0), \gamma'(0) \rangle.$$

Therefore, for any vector $X \in T\widetilde{M}$ tangent to the submanifold $S^{p,q} \times \{y_0\} \times \{t_0\} \subset \widetilde{M}$,

$$f^*g(X, X) = \cosh^2 t_0 \cdot g_1(X, X),$$

where X is considered as a tangent vector on $S^{p,q}$ in the right side.

Similar argument shows that for a vector $Y \in T\widetilde{M}$ tangent to the submanifold $\{x_0\} \times H^{r,s} \times \{t_0\} \subset \widetilde{M}$ with $t_0 \neq 0$,

$$f^*g(Y, Y) = \sinh^2 t_0 \cdot g_2(Y, Y).$$

Finally define a curve $\sigma: [0, \infty) \to \widetilde{M}$ by $\sigma(t) = (x_0, y_0, t)$, then

$$\begin{aligned} &|\, (f \circ \sigma)'(t)\, |^2 \\ =&|\, (\sinh t \cdot x_0, \cosh t \cdot y_0)\, |^2 \\ =& -1. \end{aligned}$$

Hence at $(x_0, y_0, t_0) \in \widetilde{M}$

$$f^*g = \cosh^2 t_0 \cdot g_1 + \sinh^2 t_0 \cdot g_2 - dt^2. \qquad \square$$

Note that if $s+1 \neq 0$, then f is not surjective. In fact, $S^{p+r+2,q+s+1}$ consists of three parts.

Consider projections:

$$\pi_1 : \mathbb{R}^{p+r+2,q+s+2} \to \mathbb{R}^{p+1,q+1},$$

$$\pi_2 : \mathbb{R}^{p+r+2,q+s+2} \to \mathbb{R}^{r+1,s+1}.$$

For each $z \in S^{p+r+2,q+s+1}$ let $x = \pi_1(z)$, $y = \pi_2(z)$, then we can express $z = (x,y)$ and $\langle x,x\rangle + \langle y,y\rangle = 1$.

Denote the light cones in $\mathbb{R}^{p+1,q+1}$ and $\mathbb{R}^{r+1,s+1}$ by L_1, L_2, i.e.,

$$L_1 = \{ \, x \in \mathbb{R}^{p+1,q+1} \mid \langle x,x\rangle = 0, \; x \neq 0 \, \},$$
$$L_2 = \{ \, y \in \mathbb{R}^{r+1,s+1} \mid \langle y,y\rangle = 0, \; y \neq 0 \, \}.$$

We define the three subsets of $S^{p+r+2,q+s+1}$ as follows:

$$i) \;\; \widetilde{M}_+ = \{ \, z \in S^{p+r+2,q+s+1} \mid \langle \pi_1(z), \pi_1(z)\rangle \geq 1 \, \},$$
$$ii) \;\; \widetilde{M}_0 = \{ \, z \in S^{p+r+2,q+s+1} \mid 0 \leq \langle \pi_1(z), \pi_1(z)\rangle \leq 1 \, \},$$
$$iii) \;\; \widetilde{M}_- = \{ \, z \in S^{p+r+2,q+s+1} \mid \langle \pi_1(z), \pi_1(z)\rangle \leq 0 \, \}.$$

Then $S^{p+r+2,q+s+1} = \widetilde{M}_+ \cup \widetilde{M}_0 \cup \widetilde{M}_-$.

Now we will study the structure of the three subsets. Let $\lambda, \mu \in \mathbb{R}$ and $\lambda, \mu \geq 0$.

i) If $\langle x,x\rangle = \lambda^2 > 1$ $(\lambda > 1)$, then $\langle y,y\rangle = 1 - \langle x,x\rangle = -\mu^2 < 0$, $(\mu > 0)$. Put $u = (1/\lambda)x$, $v = (1/\mu)y$, then $u \in S^{p+1,q}$, $v \in H^{r,s+1}$ and $\lambda^2 - \mu^2 = 1$. We can parametrize λ, μ by $\lambda = \cosh t$, $\mu = \sinh t$, then

$$z = (\cosh t \cdot u, \; \sinh t \cdot v), \qquad t \in (0, \infty).$$

Define a subset M_+ of $\widetilde{M}_+$ by

$$M_+ = \big\{ (\cosh t \cdot u, \sinh t \cdot v) \in \widetilde{M}_+ \mid \; u \in S^{p+1,q}, \; v \in H^{r,s+1}, t \in [0, \infty) \big\}.$$

Then the same argument as in Lemma 2.1 shows that

$$M_+ \cong \left(S^{p+1,q} \times H^{r,s+1} \times [0,\infty), \; \cosh^2 t \cdot g_1 + \sinh^2 t \cdot h_2 - dt^2 \right),$$

where '$\cong$' means 'isometric', g_1 is the metric on $S^{p+1,q}$ and h_2 is the metric on $H^{r,s+1}$. The locus $\langle x,x\rangle = 1$, $\langle y,y\rangle = 0$ $(y \neq 0)$ is $S^{p+1,q} \times L_2$. Thus $\widetilde{M}_+ = M_+ \cup (S^{p+1,q} \times L_2)$.

Similarly we define

$$M_0 = \Big\{(\cos t \cdot u, \sin t \cdot v) \in \widetilde{M_0} \mid u \in S^{p+1,q}, \\ v \in S^{r+1,s}, t \in [0, \frac{\pi}{2}]\Big\},$$

$$M_- = \Big\{(\sinh t \cdot u, \cosh t \cdot v) \in \widetilde{M}_- \mid u \in H^{p,q+1}, \\ v \in S^{r+1,s}, t \in [0,\infty)\Big\},$$

then by the same argument as in i), we get the following:

ii) $M_0 \cong (S^{p+1,q} \times S^{r+1,s} \times [0, \frac{\pi}{2}],\ \cos^2 t \cdot g_1 + \sin^2 t \cdot g_2 + dt^2)$, where g_2 is the metric on $S^{r+1,s}$ and $\widetilde{M_0} = M_0 \cup (S^{p+1,q} \times L_2) \cup (L_1 \times S^{r+1,s})$,

iii) $M_- \cong (H^{p,q+1} \times S^{r+1,s} \times [0,\infty),\ \sinh^2 t \cdot h_1 + \cosh^2 t \cdot g_2 - dt^2)$, where h_1 is the metric on $H^{p,q+1}$ and $\widetilde{M}_- = M_- \cup (L_1 \times S^{r+1,s})$.

Note that $M_+ \cap M_0 \cong S^{p+1,q}$ and $M_0 \cap M_- \cong S^{r+1,s}$ are the focal varieties.

In summary, we have

THEOREM 2.2.

$$S^{p+r+2,q+s+1} = M_+ \cup (S^{p+1,q} \times L_2) \cup M_0 \cup (L_1 \times S^{r+1,s}) \cup M_-,$$

where $M_+ \cap M_0 = S^{p+1,q} \times \{0\}$ *and* $M_0 \cap M_- = \{0\} \times S^{r+1,s}$ *are the focal varieties of the equivariant manifolds* M_+, M_0 *and* M_-.

By the same argument, we get the following useful proposition.

PROPOSITION 2.3. *Each* $z \in S^{p+r+2,q}$ *can be written as*

$$z = (\cosh t \cdot u,\ \sinh t \cdot v),$$

where $u \in S^{p+1,q}$, $v \in H^{r,0}$ *and* $t \in [0,\infty)$.

And so $S^{p+r+2,q}$ *is isometric to*

$$\big(S^{p+1,q} \times S^r \times [0,\infty),\ \cosh^2 t \cdot g - \sinh^2 t \cdot h - dt^2\big),$$

where g *is the metric on* $S^{p+1,q}$ *and* h *is the metric on* S^r.

REMARK. Similar result can be obtained by the same argument on pseudohyperbolic space. This is natural because the pseudosphere and the pseudohyperbolic space are anti-isometric.

3. Construction of harmonic maps between pseudospheres

There are several types of equivariant maps on pseudosphere and pseudohyperbolic space. To construct equivariant maps, we need the following lemma because the harmonic maps with constant energy density that are known are mearly the maps between spheres and between pseudospheres.(ref. [6], [8]) By the direct computation we can obtain the following lemma.

LEMMA 3.1. *Let $\widetilde{\psi}: E \to F$ be a harmonic map with constant energy density. Define $\psi : E \to B \times_\mu F$ by*

$$\psi(x) = (p, \widetilde{\psi}(x)),$$

where $p \in B$ is fixed. Then $e(\psi)$ is constant.
Furthermore, ψ is harmonic if and only if $d\mu(p) = 0$.

REMARK. In fact ψ is the composition of $\widetilde{\psi}: E \to F$ and the inclusion $\jmath: F \hookrightarrow B \times_\mu F$ defined by $\jmath(x) = (p, x)$. Generally the composition of two harmonic maps is not harmonic. And this lemma asserts that if $d\mu(p) = 0$ $(p \in B)$ then $\jmath: F \hookrightarrow B \times_\mu F$ and the composition $\jmath \circ \widetilde{\psi}$ are harmonic.

Using the fact that $S^{r+1,s}$ is isometric to the warped product

$$(\mathbb{R}^{r+1} \times S^s, \ -h + (1+ | y |^2)\widetilde{h}),$$

where h, and $\widetilde{h}$ are the canonical Riemannian metric on $\mathbb{R}^{r+1}$ and S^s respectively and $y \in \mathbb{R}^{r+1}$, $| y |^2 = h(y, y)$, we get

COROLLARY 3.2. *Let $\widetilde{\psi}: S^p \to S^s$ be a harmonic map with constant energy density. Define $\psi: S^p \to S^{r+1,s}$ by $\psi(x) = (0, \widetilde{\psi}(x))$ for each*

$x \in S^p$. *Then* ψ *is a harmonic map with constant energy density* $e(\psi) = e(\widetilde{\psi})$.

As in the case of Riemannian manifold, the equivariant manifolds and equivariant maps can be defined in the analoguous way in the semi-Riemannian case. And the same reduction occurs for the harmonic map equation of the equivariant map.(ref. [5])

With the aid of the above Corollary, we can construct an equivariant map. Let $\varphi: S^{p_1+1,q_1} \to S^{p_2+1,q_2}$ and $\widetilde{\psi}: S^{r_1} \to S^{s_2}$ be harmonic homogeneous polynomial maps of degree k_1, k_2. Then φ and $\widetilde{\psi}$ are harmonic maps with constant energy density:

$$e(\varphi) = \frac{\lambda_1}{2}, \quad e(\widetilde{\psi}) = \frac{\lambda_2}{2},$$

where $\lambda_1 = k_1(k_1 + p_1 + q_1)$ and $\lambda_2 = k_2(k_2 + r_2 - 1)$ (see [6] and [1]). Define $\psi: S^{r_1} \to S^{r_2+1,s_2}$ as in the Corollary then ψ is harmonic and $e(\psi) = \lambda_2/2$.

Recall that

$$S^{p_1+r_1+2,q_1} \cong \left(S^{p_1+1,q_1} \times S^{r_1} \times [0,\infty),\ \cosh^2 t \cdot g - \sinh^2 t \cdot h - dt^2\right),$$

where g and h are the metrics on S^{p_1+1,q_1} and S^{r_1}, respectively, and $M_0 \subset S^{p_2+r_2+2,q_2+s_2+1}$ is isometric to

$$\left(S^{p_2+1,q_2} \times S^{r_2+1,s_2} \times [0,\frac{\pi}{2}],\ \cos^2 t \cdot g_1 + \sin^2 t \cdot g_2 + dt^2\right),$$

where g_1 and g_2 are the metrics on S^{p_2+1,q_2} and S^{r_2+1,s_2}, respectively.

LEMMA 3.3. *Define an equivariant map* $\Phi: S^{p_1+r_1+2,q_1} \to M_0$ *by*

$$\Phi(\cosh t \cdot u, \sinh t \cdot v) = \left(\cos\alpha(t) \cdot \varphi(u),\ \sin\alpha(t) \cdot \psi(v)\right),$$

where $u \in S^{p_1+1,q_1}$, $v \in S^{r_1}$ *and* $t \in [0,\infty)$, *then there exists a function* $\alpha: [0,\infty) \to [0, \frac{\pi}{2}]$ *such that* Φ *is harmonic.*

Proof. By the reduction theorem (ref. [5]), the harmonicity equation is reduced to

$$\begin{cases} \alpha''(t) + (m_1 \tanh t + m_2 \coth t)\alpha'(t) \\ \qquad - \left(\frac{\lambda_1}{\cosh^2 t} + \frac{\lambda_2}{\sinh^2 t}\right) \sin\alpha(t)\cos\alpha(t) = 0 \\ \alpha(0) = 0, \end{cases}$$

where $m_1 = p_1 + q_1 + 1,\ m_2 = r_2$.

By the change of variable $e^s = \sinh t$, the above equation becomes

$$\begin{cases} \beta''(s) + \dfrac{1}{e^s + e^{-s}}\{(m_1+1)e^s + (m_2-1)(e^s + e^{-s})\}\beta'(s) \\ \qquad - \dfrac{1}{e^s + e^{-s}}\left(\dfrac{\lambda_1}{e^s + e^{-s}} + \dfrac{\lambda_2}{e^s}\right)\sin\beta(s)\cos\beta(s) = 0 \\ \lim\limits_{s\to-\infty}\beta(s) = 0. \end{cases}$$

By the same argument as in [1], there exists a solution of the boundary value problem. Hence the equivariant map Φ is harmonic. □

REMARK. About the regularity of Φ, similar arguments as in [1] can be applied from the continuity of Φ and the structure of the forcal variety. In fact we can see that Φ is smooth.

Now we will show that Φ defined in Lemma 3.3 is harmonic as a map between pseudospheres.

LEMMA 3.4. *Let $\mathcal{S} = S^{p+r+2,q+s+1}$ and consider $M_0 \subset \mathcal{S}$ defined in Section 2. The focal varieties of M_0,*

$$S^{p+1,q} \times \{0\} \quad \textit{and} \quad \{0\} \times S^{r+1,s},$$

are totally geodesic submanifolds in $\mathcal{S}$.

Proof. Recall that every geodesic in $\mathcal{S} \subset \mathbb{R}^{p+r+2,q+s+2}$ is the intersection of $\mathcal{S}$ and a plane Π through the origin of $\mathbb{R}^{p+r+2,q+s+2}$ (ref. [7]).

Given a geodesic γ in $S^{p+1,q} \subset \mathbb{R}^{p+1,q+1}$, there is a plane $\Pi \subset \mathbb{R}^{p+1,q+1}$ such that γ is a parametrization of $S^{p+1,q} \cap \Pi$.

Regard Π as a plane in $\mathbb{R}^{p+1,q+1} \times \mathbb{R}^{r+1,s+1}$, then we obtain

$$\begin{aligned} &\Pi \cap \left(S^{p+1,q} \times \{0\}\right) \\ =&\Pi \cap \left(\mathcal{S} \cap (\mathbb{R}^{p+1,q+1} \times \{0\})\right) \\ =&\left(\Pi \cap (\mathbb{R}^{p+1,q+1} \times \{0\})\right) \cap \mathcal{S} \\ =&\Pi \cap \mathcal{S}. \end{aligned}$$

Therefore a geodesic in $S^{p+1,q} \times \{0\}$ is a geodesic in $\mathcal{S}$ and $S^{p+1,q} \times \{0\}$ is a totally geodesic submanifold in $\mathcal{S}$.

Similarly $\{0\} \times S^{r+1,s}$ is also totally geodesic in $\mathcal{S}$. □

By the above lemmas, we can obtain a harmonic map between pseudospheres.

THEOREM 3.5. *Let* $\jmath: M_0 \hookrightarrow S^{p_2+r_2+2,q_2+s_2+1}$ *be the inclusion and*

$$\Phi: S^{p_1+r_1+2,q_1} \to M_0$$

be the harmonic map obtained in Lemma 3.3. Then the composition

$$\jmath \circ \Phi: S^{p_1+r_1+2,q_1} \to S^{p_2+r_2+2,q_2+s_2+1}$$

is harmonic.

EXAMPLE. There are several harmonic maps with constant energy density between pseudospheres and between spheres. For example, if we choose harmonic maps with constant energy density $\varphi: S^{2,1} \to S^{1,1}$ (cf. [6]) and $\widetilde{\psi}: S^3 \to S^2$ (cf. [1]), then we get harmonic maps

$$\Phi: S^{6,1} \to S^{r+2,4}$$

for any $r \geq 0$, by the above Theorem.

References

1. P. Baird, *Harmonic Maps with Symmetries, Harmonic Morphisms and Deformation of Metrics*, Research Notes in Math, Pitman, 1983.
2. J. Eells and L. Lemaire, *Another Report on Harmonic Maps,*, Bull.London Math. Soc. No. 86, Part 5, **20** (1988), 385-524.
3. J. Eells and J. H. Sampson, *Harmonic mapping of Riemannian manifolds*, Amer. J. Math., **86** (1964), 109-160..
4. J. F. Glazebrook, *Strings, Harmonic Maps and Hyperbolic Systems*, Computers Math. Applic. **19, No. 8/9** (1990), 117-125.
5. M. J. Jeon, *Equivariant Harmonic Maps between Pseudo-Riemannian Manifolds*, Ph. D. Thesis, Seoul National Univ. (1992).
6. J. J. Konderak, *Construction of Harmonic Maps between Pseudo-Riemannian Sphere and Hyperbolic Spaces,* Proc. Amer. Math. Soc. **109, No. 2** (1990), 469-476.
7. B. O'Neil, *Semi-Riemannian Geometry with Applications to Relativity,* Academic Press, 1983.

8. R. T. Smith, *Harmonic Mappings of Spheres*, Amer. J. Math. **97** (1975), 364-385.
9. A. Ratto, *Equivariant Harmonic Maps between Manifolds with Metrics of (p, q)-Signature*, Ann. Inst. Henri Poincaré **6, No. 6** (1989), 503-524.

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