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THE CONDITION NUMBERS OF TWO INTERPOLANT MATRICES

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1. Introduction

Given the scattered data (x_i, f_i), $x_i \in R^m, f_i \in R, i = 1, \ldots, n$, the interpolation problem by radial functions consists of finding an interpolant to the data of the form

$$S(x) = \sum_{i=1}^{n} a_i g_i(x)$$

where $g_i(x)$ is a radial function. In this papaer, we consider the case of $g_i(x) = \|x - x_i\|$ and $x_i \in R^1$. Then $\| \cdot \|$ becomes absolute value.

There are some methods to find an interpolant $S(x)$ using various radial functions (Franke, 1982). In 1938, Shoenberg proved that the coefficient matrix $A_{ij} = \|x_i - x_j\|$ is nonsingular if $\|\cdot\|$ is the Euclidean norm. Hence interpolation by radial functions is always possible for any set of distinct points. But a linear system determining the radial interpolant is known to be ill-conditioned for large data sets.

To decrease the condition number of the coefficient matrix arising in the linear system, we introduce different set of points $\{y_i\}$ to define the basis functions. The new basis will be of the form $g_i(x) = \|x - y_i\|$ and the new coefficient matrix $\bar{A}$ will be $\bar{A}_{ij} = \|x_i - y_j\|$. The points y_i used to define the basis functions will be called 'knots' and the points x_i where interpolation is to be carried out will be called 'nodes'.

The new coefficient matrix $\bar{A}$ becomes singular for some set of $\{y_i\}$ while $A = \|x_i - x_j\|$ is nonsingular for any set of $\{x_i\}$. The next theorem tells the position of $\{y_i\}$ where $\bar{A}$ is nonsingular.

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THEOREM (JOONSOOK LEE,1992). *$\bar{A} = \|x_i - y_j\|$ is nonsingular if and only if $y_1 \in [x_1, x_2), y_i \in (x_{i-1}, x_{i+1})$ for $i = 2, \ldots, n-2$ and $y_n \in (x_{n-1}, x_n]$.* □

We assume above position of $\{y_i\}$ throughout this paper. Now for the matrix norm for condition number, we use $\|\cdot\|_1$-norm i.e. maximum absolute column sum.

2.The Condition Number of $A = \|x_i - x_j\|$

Since explicit forms of A and A^{-1} are known, we can actually calculate the condition number of A. Let $0 = x_1 < x_2 < \cdots < x_n = 1$. We can assume this without loss of generality since a condition number is invariant in scaling.

LEMMA 2.1 (BOS AND SALCAUSCUS, 1987). *If A is defined by $A = \|x_i - x_j\|$, then its inverse is*

$$A^{-1} =$$

$$\begin{pmatrix} \frac{h_1-1}{2h_1} & \frac{1}{2h_1} & \cdots & & 0 & 0 & \frac{1}{2} \\ \frac{1}{2h_1} & -\frac{h_1+h_2}{2h_1h_2} & \frac{1}{2h_2} & 0 & \cdots & 0 & 0 \\ 0 & \frac{1}{2h_2} & -\frac{h_2+h_3}{2h_2h_3} & \frac{1}{2h_3} & 0 & \cdots & 0 \\ \vdots & & & \ddots & & & \vdots \\ 0 & & & \cdots & \frac{1}{2h_{n-2}} & -\frac{h_{n-2}+h_{n-1}}{2h_{n-2}h_{n-1}} & \frac{1}{2h_{n-1}} \\ \frac{1}{2} & 0 & 0 & \cdots & 0 & \frac{1}{2h_{n-1}} & \frac{h_{n-1}-1}{2h_{n-1}} \end{pmatrix}$$

LEMMA 2.2. *Let $|A_m| = \sum_{i=1}^{n} |A_{im}|$. Then*

$$|A_1| > |A_2| > \cdots > |A_p| \quad \text{and} \quad |A_n| > |A_{n-1}| > \cdots > |A_{p+1}|$$

where

$$\begin{cases} p = \frac{n}{2} & \text{if } n \text{ is even} \\ p = \frac{n+1}{2} & \text{if } n \text{ is odd} \end{cases}$$

Proof. It is verified by direct calculation. We supress the detail. □

Now we state the lemma concerning the norm of A and A^{-1} without proof.

LEMMA 2.3. *Let* $h_i = |x_{i+1} - x_i|$. *Then*

$$\|A\| = \max\{|A_1|, |A_n|\}$$

$$\|A^{-1}\| = \max_{1 \le i \le n-2} \left\{ \frac{1}{h_i} + \frac{1}{h_{i+1}} \right\}$$

The next theorem gives a lower bound for $\|A\|$.

THEOREM 2.4. $\|A\| \ge \frac{n}{2}$

Proof. From lemma2.2, $\|A\| = \max\{\sum_{i=1}^{n-1}(n-i)h_i, \sum_{i=1}^{n-1}(n-i) h_{n-i}\}$. Suppose $\|A\| < \frac{n}{2}$. Then $\sum_{i=1}^{n-1}(n-i)h_i < \frac{n}{2}$ and $\sum_{i=1}^{n-1}(n-i)h_{n-i} < \frac{n}{2}$. Now

$$\begin{aligned} &\sum_{i=1}^{n-1}(n-i)h_i + \sum_{i=1}^{n-1}(n-i)h_{n-i} \\ &= \sum_{i=1}^{n-1}(n-i)(h_i + h_{n-i}) = n\sum_{i=1}^{n-1} h_i = n \quad \text{since} \quad \sum_{i=1}^{n-1} h_i = 1. \end{aligned}$$

But $\sum_{i=1}^{n-1}(n-i)h_i + \sum_{i=1}^{n-1}(n-i)h_{n-i} < \frac{n}{2} + \frac{n}{2} = n$. This is a contradiction. □

If $\sum_{i=1}^{n-1}(n-i)h_i = \sum_{i=1}^{n-1}(n-i)h_{n-i}$, the minimum condition number can be achieved. We can also show that if $\sum_{i=1}^{n-1}(n-i)h_i$ is not equal to $\sum_{i=1}^{n-1}(n-i)h_{n-i}$, then $\|A\| > \frac{n}{2}$. Indeed let $\sum_{i=1}^{n-1}(n-i)h_i > \sum_{i=1}^{n-1}(n-i)h_{n-i}$. Then $\|A\| = \sum_{i=1}^{n-1}(n-i)h_i$ and $2\|A\| = 2\sum_{i=1}^{n-1}(n-i)h_i > \sum_{i=1}^{n-1}(n-i)h_i + \sum_{i=1}^{n-1}(n-i)h_i = n\sum_{i=1}^{n-1} h_i = n$. One position which satisfying $\sum_{i=1}^{n-1}(n-i)h_i = \sum_{i=1}^{n-1}(n-i)h_{n-i}$ is equally spaced nodes.

Now we will achieve a lower bound for $\|A^{-1}\|$. But we need to put a restriction on the number of nodes. We get a lower bound for $\|A^{-1}\|$ only for the case of odd number of nodes. From now on, we let k be an index such that $\|A^{-1}\| = \frac{1}{h_k} + \frac{1}{h_{k+1}}$.

LEMMA 2.5. *If* $\|A^{-1}\| < 2(n-1)$, *then* $h_i + h_{i+1} > \frac{2}{n-1}$ *for all* i.

Proof. Suppose there exist some j such that $h_j + h_{j+1} \le \frac{2}{n-1}$. Now $(h_j + h_{j+1})^2 \ge 4h_j h_{j+1}$ and this implies that $\frac{h_j+h_{j+1}}{h_j h_{j+1}} \ge \frac{4}{h_j+h_{j+1}}$. Then by our assumption

$$\frac{h_j + h_{j+1}}{h_j h_{j+1}} = \frac{1}{h_j} + \frac{1}{h_{j+1}} \ge 2(n-1) \quad \square$$

THEOREM 2.6. $\|A^{-1}\| \ge 2(n-1)$ *for odd number of nodes.*

Proof. Suppose $\|A^{-1}\| < 2(n-1)$. Then by lemma 2.4, $h_i + h_{i+1} > \frac{2}{n-1}$ for all i. Then

$$1 = \sum_{i=1}^{n-1} h_i = (h_1 + h_2) + \cdots + (h_{n-2} + h_{n-1}) > \frac{2}{n-1} \cdot \frac{n-1}{2} = 1 \quad \square$$

THEOREM 2.7. *If* n *is odd and* $\|A^{-1}\| = 2(n-1)$, *then* $h_i = \frac{1}{n-1}$ *for all* i.

Proof. First we will show that $h_i + h_{i+1} \ge \frac{2}{n-1}$ for all i. Suppose there exist some j such that $h_j + h_{j+1} < \frac{2}{n-1}$. Then by lemma2.4, $\frac{1}{h_j} + \frac{1}{h_{j+1}} > 2(n-1)$ and this contradicts our assumption.

Now the facts $h_i + h_{i+1} \ge \frac{2}{n-1}$ for all i and $(h_1 + h_2) + \cdots + (h_{n-2} + h_{n-1}) = 1$ imply that $h_i + h_{i+1} = \frac{2}{n-1}$ for odd i. Let m be any arbitrary odd index. Then $h_m + h_{m+1} = \frac{2}{n-1}$. But by the proof of lemma2.4, $\frac{h_m+h_{m+1}}{h_m h_{m+1}} \ge 2(n-1)$ and equality holds only when $h_m = \frac{1}{n-1}$ and $h_{m+1} = \frac{1}{n-1}$. $\square$

THEOREM 2.8. *Let* n *be an odd number.* $\|A\|\|A^{-1}\| = n(n-1)$ *if and only if nodes are equally spaced.*

Proof. If $h_i = \frac{1}{n-1}$, it is obvious that $\|A\|\|A^{-1}\| = n(n-1)$. Now suppose $\|A\|\|A^{-1}\| = n(n-1)$. Since $\|A\| \ge \frac{n}{2}$ and $\|A^{-1}\| \ge 2(n-1)$, $\|A\|\|A^{-1}\| = n(n-1)$ implies that $\|A\| = \frac{n}{2}$ and $\|A^{-1}\| = 2(n-1)$. Then by theorem2.7, $h_i = \frac{1}{n-1}$. $\square$

We have shown that if n is odd, the minimum condition number interpolation matrix is $n(n-1)$. If n gets larger than 100, the condition number becomes more than 100^2. We introduce new set of points $\{y_i\}$ called 'knots' for basis and compare the two condition numbers of A and $\bar{A}$.

3. The Comparison of condition numbers of two coefficient matrices

LEMMA 3.1(JOONSOOK LEE, 1992). *If y_m is in the interval $[x_k, x_{k+1}]$, then the m-th column of $\bar{A}$ is a linear and convex combination of the k-th and the $(k+1)$-st columns of A i.e.*

$$\bar{A}_m = \left(\frac{x_{k+1} - y_m}{h_k}\right) A_k + \left(\frac{y_m - x_k}{h_k}\right) A_{k+1} \quad \square$$

Since every element of A is nonnegative, $|\bar{A}_i|$ is a linear and convex combination of $|A_j|$ and $|A_{j+1}|$ for some j.

THEOREM 3.2. $\|\bar{A}\| = \max\{|\bar{A}_1|, |\bar{A}_n|\}$.

Proof. Since we assume that $y_1 \in [x_1, x_2)$ and $y_i \in (x_{i-1}, x_{i+1})$ for $i = 2 \ldots n-1$ and $y_n \in (x_{n-1}, x_n]$, it is clear that $|\bar{A}_1|$ is a linear and convex combination of $|A_1|$ and $|A_2|$.

Let p be the same p in lemma 2.2. Then for $2 \leq i \leq p$, $|\bar{A}_i|$ is a linear and convex combination of $|A_i|$ and $|A_{i+1}|$, or $|A_i|$ and $|A_{i-1}|$. If $i \geq 3$, $|\bar{A}_1| > |\bar{A}_i|$. But if $i = 2$, $|\bar{A}_2|$ can be a linear combination of $|A_1|$ and $|A_2|$. We will compare $|\bar{A}_1|$ and $|\bar{A}_2|$.

(1)

$$\begin{aligned}
&|\bar{A}_1| - |\bar{A}_2| \\
&= \left(\frac{a_1}{h_1}|A_1| + \left(1 - \frac{a_1}{h_1}\right)|A_2|\right) - \left(\left(\frac{b_1}{h_1}\right)|A_1| + \left(1 - \frac{b_1}{h_1}\right)|A_2|\right) \\
&= \left(\frac{b_1}{h_1} - \frac{a_1}{h_1}\right)|A_1| - \left(\frac{b_1}{h_1}\right)|A_1| + \left(1 - \frac{b_1}{h_1}\right)|A_2| \\
&= \left(\frac{a_1 - b_1}{h_1}\right)(|A_1| - |A_2|)
\end{aligned}$$

where $b_1 = x_2 - y_2$ and $a_1 = x_2 - y_1$.

But $a_1 > b_1$ and (1) become positive. It follows similar analysis for $p+1 \leq i \leq n$. $\square$

COROLLARY 3.3. $\|\bar{A}\| \leq \|A\|$.

proof. It follows theorem3.2. □

Before we state the theorem about $\bar{A}^{-1}$, we need to introduce some notations by letting $e_i = \frac{|x_i - y_i|}{h_i}$ and $d_i = \frac{|x_i - y_i|}{h_{i-1}}$. Also we let R_i and $\bar{R}_i$ be the i-th rows of A and $\bar{A}$ respectively.

THEOREM 3.4 (JOONSOOK LEE, 1992). *Let $y_1 \in [x_1, x_2), y_i \in (x_{i-1}, x_{i+1})$ for $i = 2, \ldots n-1$ and $y_n \in (x_{n-1}, x_n]$. Let (x_k, x_{k+1}) be the unique subinterval containing two knots. If we assume that y_m belongs to the separated interval $[x_{p+1}, x_q]$, then*

$$\bar{R}_m = \begin{cases} \frac{1}{1-e_m} S_m & \text{if } p+1 \leq m \leq k-1 \\ \left(\frac{1-d_{k+1}}{1-e_k-d_{k+1}}\right) S_k - \left(\frac{d_{k+1}}{1-e_k-d_{k+1}}\right) T_{k+1} & \text{if } m = k \\ \left(\frac{1-e_k}{1-e_k-d_{k+1}}\right) T_{k+1} - \left(\frac{e_k}{1-e_k-d_{k+1}}\right) S_k & \text{if } m = k+1 \\ \frac{1}{1-d_m} T_m & \text{if } k+2 \leq m \leq q \end{cases}$$

where

$$S_m = \sum_{i=0}^{m-(p+1)} (-1)^{-i} a_{m-i}^{m-1} R_{m-i}$$

$$T_m = \sum_{i=0}^{q-m} (-1)^i b_{m+1}^{m+i} R_{m+i}$$

and

$$a_s^t = \begin{cases} \dfrac{e_s \ldots e_t}{(1-e_s)\ldots(1-e_t)} \\ 1 \text{if } s > t \end{cases} \quad b_s^t = \begin{cases} \dfrac{d_s \ldots d_t}{(1-e_s)\ldots(1-e_t)} \\ 1 \text{if } s > t \end{cases}$$

Since $\bar{A}$ is nonsingular with our assumption, any row of $\bar{A}$ is a linear combination of A_{ij}^{-1}. This means that $\bar{A}_{mj}^{-1}$ is a linear combination of A_{ij}^{-1} for $i = 1 \ldots n$. Now we want to compare $\|\bar{A}^{-1}\|$ and $\|A^{-1}\|$. But

$\|A^{-1}\| = \max_{1\leq i\leq n-2}\left\{\frac{1}{h_i}+\frac{1}{h_{i+1}}\right\}$ where $\frac{1}{h_i}+\frac{1}{h_{i+1}}$ is a $(i+1)$-st absolute column sum. Then we need to compare $|A_j^{-1}|$ and $|\bar{A}_j^{-1}|$ only for $j = 2\ldots n-1$ By lemma2.1, $A_{ij}^{-1} = 0$ for $i \neq j-1, j, j+1$ and $\bar{A}_{mj}^{-1}$ becomes a linear combination of $A_{j-1,j}^{-1}, A_{j,j}^{-1}, A_{j+1,j}^{-1}$. Now by theorem3.4, we can actually calculate the coefficients for $A_{j-1,j}^{-1}, A_{j,j}^{-1}, A_{j+1,j}^{-1}$. Since these coefficients depend on m and j, we denote them by $E_j(m)$. Then we write

$$\begin{aligned}\bar{A}_{mj}^{-1} =& E_{j-1}(m)A_{j-1,j}^{-1} + E_j(m)A_{j,j}^{-1} + \\ & E_{j+1}(m)A_{j+1,j}^{-1}\end{aligned}$$

Here $E_j(m)$ represents the coefficient of R_j when we write $\bar{R}_m$ as a linear combination of $R_i's$ by theorem3.4. For example $E_m(m)$ is one of $\left\{\frac{1}{1-e_m}, \frac{1-d_{m+1}}{1-e_m-d_{m+1}}, \frac{1-e_{m-1}}{1-e_{m-1}-d_m}, \frac{1}{1-d_m}\right\}$.

LEMMA 3.5. *$E_i(i) > 1, E_{i+1}(i) \leq 0, E_{i-1}(i) \leq 0, E_{i+2}(i) \geq 0$ and $E_{i-2}(i) \geq 0$.*

Proof. It is verified by direct calculation. □

THEOREM 3.6. $\|\bar{A}^{-1}\| \geq \|A^{-1}\|$.

Proof. We will compare $|\bar{A}_{m,j}^{-1}|$ with $|A_{m,j}^{-1}|$ for $j \geq 2$. By lemma2.1, $A_{m,j}^{-1} = 0$ if $m \neq j-1, j, j+1$. Then $|\bar{A}_{m,j}^{-1}| \geq |A_{m,j}^{-1}|$ for $m \neq j-1, j, j+1$. Hence we only need to consider the case $m = j-1, j, j+1$. Now

$$\bar{A}_m^{-1} j = E_{j-1}(m)A_{j-1,j}^{-1} + E_j(m)A_{j,j}^{-1} + E_{j+1}(m)A_{j+1,j}^{-1}$$

and $A_{j-1,j}^{-1}$ and $A_{j+1,j}^{-1}$ are positive and $A_{j,j}^{-1}$ is negative.

First we consider $m = j-1$.

$$\bar{A}_{j-1,j}^{-1} = E_{j-1}(j-1)A_{j-1,j}^{-1} + E_j(j-1)A_{j,j}^{-1} + E_{j+1}(j+1)A_{j+1,j}^{-1}.$$

Then by lemma3.5, all three terms are positive and

$$\begin{aligned}&|\bar{A}_{j-1,j}^{-1}| \\ =&|E_{j-1}(j-1)A_{j-1,j}^{-1}| + |E_j(j-1)A_{j,j}^{-1}| + |E_{j+1}(j-1)A_{j+1,j}^{-1}| \\ \geq&|A_{j-1,j}^{-1}|\end{aligned}$$

The cases $m = j, j + 1$ follow similar analysis. □

By corollary3.3, we know that $\|\bar{A}\|$ is always smaller than $\|A\|$. But only y_1 and y_n may contribute to improve $\|\bar{A}\|$ since $\|\bar{A}\| = \max\{|\bar{A}_1|, |\bar{A}_n|\}$. On the other hand, $\|A^{-1}\|$ is getting bigger as more knots are picked. We may improve the condition number over all if we improve $\|\bar{A}\|$ while holding $\|\bar{A}^{-1}\|$ egual to $\|A^{-1}\|$.

References

1. Bos,L.P. and Salkauskas,K., *On the matrix* $[|x_i - x_j|^3]$ *and the cubic spline continuity equations*, Journal of Approximation Theory **51** (1987), 81-88.
2. Franke, *Scattered Data Interpolation: tests of some methods*, Math.Comp. **38** (1982), 181-200.
3. Lee, Joonsook, *Singularity of a Coefficient Matrix* Applied Functional Analysis and Optimization, (to appear).
4. Schoenberg,I.J., *Metric Spaces and Compeletly Monotone Functions*, Ann. of Math. **38** (1938), 811-841.

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