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THE DECOMPOSITION OF ALL SMOOTH VECTOR FIELDS ON $SU(3)/T(r,s)$ INTO IRREDUCIBLE UNITARY REPRESENTATIONS

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0. Introduction.

Applying Frobenius' reciprocity law (cf. Proposition 2) and Urakawa's theorem (cf. Proposition 1), we completely decompose the set $\mathfrak{X}(M(r,s))$ of all smooth vector fields on $SU(3)/T(r,s)$ into irreducible unitary representiations.

1. Preliminaries and Main Results.

1.1. In this section, we present some results on irreducible unitary representations of a compact connected Lie group. Throughout this section we use the following notation.

G : a compact connected Lie group; G_o: the commutator subgroup of G; T(resp. T_o): a maximal toral subgroup of G (resp. G_o); $\mathfrak{g}$(resp. $\mathfrak{g}_o, \mathfrak{t}, \mathfrak{t}_o$): the Lie algebra of G (resp. G_o, T, T_o); $\mathfrak{g}_o^C$ (resp. $\mathfrak{t}_o^c$): the complexification of $\mathfrak{g}_o$(resp. $\mathfrak{t}_o$).

We choose a positive definite inner produdct $<\ ,\ >$ on $\mathfrak{g}$ which is invariant under $Ad(G)$, where Ad denotes the adjoint representation of G. Fixing a lexicographic order $>$ in $\sqrt{-1}\ \mathfrak{t}_o^*$, let P be the set of all positive roots of $\mathfrak{g}_o^c$ ralative to $\mathfrak{t}_o^c$. We denote by δ half the sum of all elements in P. Let $\Gamma(G) = \{H \in \mathfrak{t} \mid exp(H) = e\}$ and $I = \{\lambda \in \sqrt{-1}\ \mathfrak{t}^* \mid \lambda(H) \in \sqrt{-1}\ 2\pi Z$ for all $H \in \Gamma(G)\}$. An element in I is called a G-integral form. The elements of

$$D(G) = \{\lambda \in I; \langle\lambda,\ \alpha\rangle\ \geq 0 \text{ for all } \alpha \in P\}$$

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are called dominant G-integral forms. Then there exists a natural bijection from $D(G)$ onto the set $\hat{G}$ of all nonequivalent finite dimensional irreducible unitary representations of G which map a dominant G-integral form $\lambda \in D(G)$ to an irreducible unitary representation (V_λ, Π_λ) having highest weight λ. For $\lambda \in D(G)$, put $d(\lambda)$ the dimension of the representation of (V_λ, Π_λ). $d(\lambda)$ is given by

$$d(\lambda) = \prod_{\alpha \in P} \frac{\langle \lambda + \delta, \alpha \rangle}{\langle \delta, \alpha \rangle}. \tag{1}$$

1.2. We consider the following 7-dimensional homogeneous space $M(r,s)$
$:= SU(3)/T(r,s)$ adimitting positively curved Riemannian metrics, which was discovered by S. Aloff and N.R. Wallach (cf. [1]).

We preserve the notation used in 1.1. Let $G = SU(3)$ and $\mathfrak{g} = \mathfrak{su}(3)$, the Lie algebra of $SU(3)$, and let $T := T(r,s) := \{diag[e^{2\pi ir\theta}, e^{2\pi is\theta}, e^{-2\pi i(r+s)\theta}]; \theta \in R\}$, $|r| + |s| \neq 0$ $(r,s \in Z)$, $i = \sqrt{-1}$. Here $diag[x,y,z]$ denotes a diagonal matrix of order 3 whose diagonal entries are x, y and z. Consider the coset manifold $M(r,s)$ which is simply connected and $H^4(M(r,s), Z) \cong Z/cZ$ with $c = r^2 + rs + s^2$, provied r, s are relatively prime. The Lie algebra $\mathfrak{t}(r,s)$ of $T(r,s)$ is included in a maximal abelian subalgebra $\mathfrak{t}$ of $\mathfrak{g}$ given by

$$\mathfrak{t} = \{diag[x_1, x_2, x_3];\ x_j \in R\ (j = 1,2,3),\ x_1 + x_2 + x_3 = 0\}.$$

We choose an inner product $< \ , \ >$ which is defined by Killing form B on $\mathfrak{g}$, i.e.,

$$\langle X, Y \rangle := -B(X,Y) = -6\ Trace(XY),\ (X, Y \in \mathfrak{g}). \tag{2}$$

Let g be the G-invariant Riemannian metric on $M(r,s)$ induced from this inner product $< \ , \ >$, and let $\mathfrak{X}(M(r,s))$ be the set of all C^∞-vector fields on $M(r,s)$. We define an inner product $((\ , \ ))$ on $\mathfrak{X}(M(r,s))$ by

$$((X,Y)) := \int_{M(r,s)} g(X,Y) v_g, \tag{3}$$

and define the Hermitian inner product $((\ , \ ))$ on $\mathfrak{X}^C(M(r,s))$ which is the complexification of $\mathfrak{X}(M(r,s))$. The translation $\tau_x, (x \in G)$, of $M(r,s)$ is defined by $\tau_x : M(r,s) \ni \bar{y} \to \overline{xy} \in M(r,s)$. Then $(\tau, \mathfrak{X}^C(M(r,s)))$ is a unitary representation of G.

1.3. We preserve the notation used in 1.2. The G-action on $\mathfrak{X}^C(M(r,s))$ is defined by

$$(4) \qquad \big((\tau_x)_* V\big)_y := (\tau_x)_* V_{\overline{x^{-1}y}}, \quad x, y \in G, \quad V \in \mathfrak{X}\big(M(r,s)\big).$$

In this paper, we get the following Main Theorem and Corollary.

THEOREM. *Let $(\tau, \mathfrak{X}^C(M(r,s))) = \sum_{\lambda \in D(SU(3))} m(\lambda) V_\lambda$ be the decomposition of $\mathfrak{X}^C(M(r,s))$ into irreducible unitary representations of $SU(3)$. Assume r and s are relatively prime. Then $D(SU(3))$, $d(\lambda) = dim_C V_\lambda$ and $m(\lambda)$ are as follows:*

1) $D(SU(3)) = \{\lambda = m_1 e_1 + m_2 e_2 \mid m_1 \geq m_2 \geq 0, m_j \in Z (j = 1,2)\}$,

2) $d(\lambda) = \frac{1}{2}(m_1 - m_2 + 1)(m_1 + 2)(m_2 + 1)$,
$\lambda = m_1 e_1 + m_2 e_2 \in D(SU(3))$,

3) *For $\lambda = m_1 e_1 + m_2 e_2 \in D(SU(3))$, if $(m_1 + m_2)$ is a multiple of 3*

1. in case of $m_2 = 0, 1, 2$,

$$\begin{cases} m_1 & = & 0 & 3n & 2 & 2+3n & 4 & 4+3n \\ m_2 & = & 0 & 0 & 1 & 1 & 2 & 2 \\ m(\lambda) & \geq & 1 & 7 & 8 & 14 & 15 & 21 \end{cases}$$

2. in case of $m_2 = 3n$,

$$m(\lambda) \geq \begin{cases} 21m + 7 & \text{when } m_1 = 3n + 3m, \ (m = 0, 1, 2, \cdots, n-1), \\ 21n + 1 & \text{when } m_1 = 6n, \\ 21n + 7 & \text{when } m_1 = 6n + 3m, \ (m = 1, 2, \cdots), \end{cases}$$

3. in case of $m_2 = 3n + 1$,

$$m(\lambda) \geq \begin{cases} 21m + 14 & \text{when } m_1 = 3n + 3m + 2, \\ & (m = 0, 1, 2, \cdots, n-1), \\ 21n + 8 & \text{when } m_1 = 6n + 2, \\ 21n + 14 & \text{when } m_1 = 6n + 3m + 2, (m = 1, 2, \cdots), \end{cases}$$

4. in case of $m_2 = 3n+2$,

$$m(\lambda) \geq \begin{cases} 21(m+1) & \text{when } m_1 = 3n+3m+4, \\ & (m = 0,1,2,\cdots,n-1), \\ 21n+15 & \text{when } m_1 = 6n+4, \\ 21(n+1) & \text{when } m_1 = 6n+3m+4, (m = 1,2,\cdots), \end{cases}$$

where , in each case of 3), n varies over the set of all the natural numbers.

COROLLARY. *The G-irreducible representation V_λ with highst weight $\lambda = 0$ contained in $\mathfrak{X}^C(M(r,s))$ is given by $\{fX_7\}_C$=$\{f \otimes X_7\}_C$, where f is a constant function on G and X_7 is an orthogonal element to $\mathfrak{t}(r.s)$ in $\mathfrak{t}$ with respect to $<\ ,\ >$.*

2. Proof of the Main Results

Following the notations used in 1.2 and 1.3, we will prove the Main Results. The Lie algebra $\mathfrak{sl}_3(C)$ of $SL_3(C)$ is the complexification of the real Lie algebra $\mathfrak{su}(3)$ of $SU(3)$. Let E_{ij} denote a square matrix with the (i,j)-entry being 1, and all the other entries being 0. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{sl}_3(C)$ which consists of the diagonal matrices of trace 0. Then we have the direct sum decomposition

$$\mathfrak{sl}_3(C) = \mathfrak{h} + \sum_{i \neq j} CE_{ij}. \tag{5}$$

Then we have

$$[H, E_{ij}] = \big(e_i(H) - e_j(H)\big)E_{ij}, \quad (H \in \mathfrak{h}). \tag{6}$$

Hence the non-zero roots of $sl_3(C)$ with respect to $\mathfrak{h}$ are

$$e_i - e_j,\ (1 \leq i,j \leq 3,\ i \neq j). \tag{7}$$

We put

$$\alpha := e_1 - e_2,\ \beta := e_2 - e_3,\ \gamma := e_1 - e_3. \tag{8}$$

We fix an order on $\sqrt{-1}\mathfrak{t}^*$ in such a way that $\alpha > \beta > 0$. On the other hand, the elements $H_{e_i - e_j} \in \sqrt{-1}\mathfrak{t}$ such that $(e_i - e_j)(H) = B(H_{e_i-e_j}, H)$ for all $H \in \mathfrak{t}^C$ are given as follows.

(9)
$$\begin{cases} H_\alpha = diag[1/6, -1/6, 0], & H_\beta = diag[0, 1/6, -1/6], \\ H_\gamma = diag[1/6, 0, -1/6]. \end{cases}$$

Following the order $>$ on $\sqrt{-1}\mathfrak{t}^*$, we get

(10)
$$P = \{\alpha, \beta, \gamma\}, \quad \delta = 2e_1 + e_2.$$

Then the set $D(G)$ of all dominant integral forms on G relative to $\mathfrak{t}$ is given by

(11)
$$D(G) = \{\lambda = m_1 e_1 + m_2 e_2 |\ m_1 \geq m_2 \geq 0,\ and\ m_j \in Z (j = 1, 2)\}.$$

For $\lambda = m_1 e_1 + m_2 e_2 \in D(G)$, we get from (1), (9) and (10)

(12)
$$d(\lambda) = \frac{1}{2}(m_1 - m_2 + 1)(m_1 + 2)(m_2 + 1).$$

We identify $\mathfrak{X}(M(r,s))$ with the following $C_T^\infty(G, \mathfrak{m})$ in the following definition (cf.[3]). Here $\mathfrak{m}$ is the orthogonal complement of $\mathfrak{t}(r,s)$ in $\mathfrak{g}$.

DEFINTION. Let $C^\infty(G, \mathfrak{m})$ be the space of all smooth maps of G into $\mathfrak{m}$. We define the subspace $C_T^\infty(G, \mathfrak{m})$ of $C^\infty(G, \mathfrak{m})$ by

(13)
$$C_T^\infty(G, \mathfrak{m}) := \{f \in C^\infty(G, \mathfrak{m}); f(xh) = Ad(h^{-1})f(x), x \in G, h \in T\}.$$

The identification Φ of $\mathfrak{X}(M(r,s))$ with $C_T^\infty(G, \mathfrak{m})$,
$\Phi; C_T^\infty(G, \mathfrak{m}) \to \mathfrak{X}(M(r,s))$, is given by

(14)
$$\Phi(f)(\bar{x}) := (\tau_x)_*\big(f(x)\big)_o, \ x \in G.$$

Here $X_o, (X \in \mathfrak{m})$, is the tangent vector of $M(r,s)$ at the origin $o := \{T\}$ which is defined by $X_o f = d/dt\ f(exp\ tX\dot{o})|_{t=0}$, and $(\tau_x)_*$ is the differential of the translation τ_x of $M(r,s)$. Under the G-action on $C_T^\infty(G, \mathfrak{m})$ defined by

(15)
$$(\tau_x f)(y) := f(x^{-1}y), \ x, y \in G, \ f \in C_T^\infty(G, \mathfrak{m}),$$

Φ is a G-isomorphism, that is,

$$(16) \qquad (\Phi \circ \tau_x)(f) = \big((\tau_x)_* \circ \Phi\big)(f), \ x \in G, \ f \in C_T^\infty(G, \mathfrak{m}).$$

For every $m \in Z$, the following homomorphism $\mathcal{X}_m$ of the 1-dimensional group T into the multiplicative group $\{z \in C; |z| = 1\}$ is well defined:

$$\mathcal{X}_m : T \ni diag\left[e^{2\pi i r\theta}, e^{2\pi i s\theta}, e^{-2\pi i(r+s)\theta}\right] \to e^{2\pi i m\theta}, \ i = \sqrt{-1}.$$

Hence $\mathcal{X}_m (m \in Z)$ are characters of $T(r, s)$. In fact, we have

$$diag\left[e^{2\pi i r\theta}, e^{2\pi i s\theta}, e^{-2\pi i(r+s)\theta}\right] = \text{identity} \iff \theta \in Z,$$

since r, s are relatively prime.

To compute $m(\lambda)$, we apply the following two propositions.

PROPOSITION 1 ([6]). *Assume r and s in T are relatively prime. Let (V_λ, π_λ) be an irreducible unitary representation of G with the highest weight $\lambda = m_1e_1 + m_2e_2 \in D(G)$. Then, as a representation of T, V_λ is decomposed into T-irreducible submodules as follows:*

$$(17) \qquad V_\lambda = \sum_{p=m_2+1}^{m_1+1} \sum_{q=0}^{m_2} \sum_{d=0}^{p-q-1} V_{r(m_1+m_2+2-2p-q+d)+s(1-p+q+2d)},$$

where $V_m (m \in Z)$ is the 1-dimensional irreducible T-submodule of V_λ with the character $\mathcal{X}_m$.

PROPOSITION 2. *(Frobenius' reciprocity Theorem [2, 3]). The multiplicity $m(\lambda)$ of V_λ, $\lambda \in D(G)$, in $C_T^\infty(G, \mathfrak{m}^C)$ is*

$$dimHom_G(V_\lambda, C_T^\infty(G, \mathfrak{m}^C)) = dimHom_T(V_\lambda, \mathfrak{m}^C),$$

where $\mathfrak{m}^c$ is $Ad(T)$-module.

By above two the propositions, we get for $\lambda \in D(G)$

$$(18) \qquad m(\lambda) = \begin{cases} \text{the number of elements } m, (m \text{ in } V_m \text{ of} \\ \text{the right side of (17)), which belong to} \\ \{\pm(k-l), \ \pm(2k+l), \ 0, \ \pm(k+2l)\ \}. \end{cases}$$

We evaluate the number $m(\lambda)$, $\lambda = m_1e_1 + m_2e_2 \in D(G)$, by using a computer. In the table below, we express $n(\lambda)$ by the number whose position is (m_2, m_1) (see Fig.1). Then, $n(\lambda) \leq m(\lambda)$.

Thus we obtain the desired result for $m(\lambda)$ of the main theorem.

Finally, we define $(C^\infty(G)\otimes\mathfrak{m})_T$ by the subspace of $C^\infty(G)\otimes\mathfrak{m}$ consisting all elements $\sum_{i=1}^{l} f_i \otimes Y_i \in C^\infty(G)\otimes\mathfrak{m}$ satisfying $\sum_{i=1}^{l} R_h f_i \otimes Ad(h)Y_i = \sum_{i=1}^{l} f_i \otimes Y_i$ for all $h \in T$, where $(R_h f)(x) := f(xh)$ $(h \in T, x \in G, f \in C^\infty(G))$.

Under the G-action on $C^\infty(G)\otimes\mathfrak{m}, (\tau_x f)(y) := f(x^{-1}y), \tau_x(f \otimes X) := \tau_x f \otimes X, (x, y \in G,\ f \in C^\infty(G),\ X \in \mathfrak{m})$, the $(C^\infty(G)\otimes\mathfrak{m})_T$ is a G-submodule. Then, $C^\infty_T(G,\mathfrak{m})$ and $(C^\infty(G)\otimes\mathfrak{m})_T$ are G-isomorphic by correspondence $f \to \sum_{j=1}^{7} f_j \otimes X_j$, where $f(x) = \sum_{j=1}^{7} f_j(x)X_j, x \in G$ and $(X_j)_{j=1}^{7}$ is an orthonormal basis of $\mathfrak{m}$ such that X_7 is an orhtogonal element to $\mathfrak{t}(k,l)$ in $\mathfrak{t}$ with respect to $<\ ,\ >$.

From the above correspondences and main theorem, the corollary is obtained.

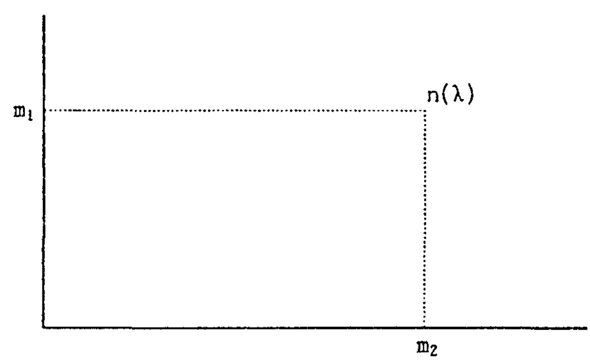

Fig. 1

Table 1

35		14			35			56			77			98		
34			21			42			63			84			105	
33	7			28			49			70			91			112
32		14			35			56			77			98		
31			21			42			63			84			105	
30	7			28			49			70			91			106
29		14			35			56			77			98		
28			21			42			63			84			99	
27	7			28			49			70			91			91
26		14			35			56			77			92		
25			21			42			63			84			84	
24	7			28			49			70			85			70
23		14			35			56			77			77		
22			21			42			63			78			63	
21	7			28			49			70			70			49
20		14			35			56			71			56		
19			21			42			63			63			42	
18	7			28			49			64			49			28
17		14			35			56			56			35		
16			21			42			57			42			21	
15	7			28			49			49			28			7
14		14			35			50			35			14		
13			21			42			42			21				
12	7			28			43			28			7			
11		14			35			35			14					
10			21			36			21							
9	7			28			28			7						
8		14			29			14								
7			21			21										
6	7			22			7									
5		14			14											
4			15													
3	7			7												
2		8														
1																
0	1															
m_1 / m_2	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15

Table 2

51			133			154			175			175			154	
50	119			140			161			176			161			140
49		126			147			168			168			147		
48			133			154			169			154			133	
47	119			140			161			161			140			119
46		126			147			162			147			126		
45			133			154			133			112			91	
44	119			140			155			140			133			112
43		126			147			147			126			105		
42			133			148			133			112			91	
41	119			140			140			119			98			77
40		126			141			126			105			84		
39			133			133			112			91			70	
38	119			134			119			98			77			56
37		126			126			105			84			63		
36			127			112			91			70			49	
35	119			119			98			77			56			35
34		120			105			84			63			42		
33			112			91			70			49			28	
32	113			98			77			56			35			14
31		105			84			63			42			21		
30			91			70			49			28			7	
29	98			77			56			35			14			
28		84			63			42			21					
27			70			49			28			7				
26	77			56			35			14						
25		63			42			21								
24			49			28			7							
23	56			35			14									
22		42			21											
21			28			7										
20	35			14												
19		21														
18			7													
17	14															
16																
m_1 / m_2	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31

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