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STEIN–NESS OF OPEN SUBSETS IN COMPLEX SPACES OF FINITE DIMENSION

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1. Introduction

Let X be a complex space of finite dimension. If Ω_1 and Ω_2 are open Stein subsets of X then $\Omega_1 \cup \Omega_2$ is not necessarily Stein. If $A_1 \subset A_2 \subset \dots$ be a sequence of open Stein subsets in a Stein manifold X then $\Omega = \bigcup_{j=1}^{\infty} A_j$ is Stein. If X is a Stein space, it is not known whether Ω should be Stein. In 1976 J. E. Fornæss[6] has given an example of a sequence of increasing Stein subsets in a manifold whose union is not Stein. In 1985 L.M.Tovar[14] proved the following theorm. The proof given there is based on the techniques of E.Ballico[1].

THEOREM 1.1([14]). *Let Y_1 and Y_2 be open Stein subsets of a holomorphically separable complex space X of finite dimension, and let $Y = Y_1 \cup Y_2$. If $dim_{\mathbb{C}} H^1(Y, \mathcal{O}) < \infty$ for the structure sheaf $\mathcal{O}$ of X and X is Stein or $Y \Subset X$, then $Y$is Stein.*

Now, we investigate that an open subset having a finite simple chain Stein cover is a Stein space if its first cohomology group is finite dimensional.

2. Preliminaries

DEFINITION 2.1. Let X and $\mathcal{F}$ be topological spaces (not necessarily Hausdorff spaces). $\mathcal{F}$ is called a sheaf on X if the following conditions are satisfied:

(1) $\pi : \mathcal{F} \longrightarrow X$ is a local homeomorphism.

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(2) For each $x \in X$, the set $\mathcal{F}_x := \pi^{-1}(x)$ has the structure of an abelian group.
(3) The group operations are continuous in the topology of $\mathcal{F}$.

Let X be a (paracompact) complex space and $\mathcal{U} = \{U_j : j \in I\}$ be a locally finite open covering of X. We define the p-th cohomology group $H^p(X, \mathcal{F})$ of X with coefficients in $\mathcal{F}$ as $H^p(X, \mathcal{F}) = \lim_{\mathcal{U}} H^p(\mathcal{U}, \mathcal{F})$. Then we have a natural homomorphism

$$\lambda^p : H^p(\mathcal{U}, \mathcal{F}) \longrightarrow H^p(X, \mathcal{F}).$$

LEMMA 2.2. *If X has a Stein open covering $\mathcal{U}$ and $\mathcal{F}$ is a coherent analytic sheaf on X, then λ^1 is bijective and λ^2 is injective.*

Proof. Let $\mathcal{U} = \{U_j : j \in I\}$ be the Stein open covering of X. Since each U_j is Stein, we have $H^1(U_j, \mathcal{F}) = 0$ by the theorem of Cartan-Oka-Serre. By Thoerem 3.5 of K.Kodaira[11], we have $H^1(\mathcal{U}, \mathcal{F}) = H^1(X, \mathcal{F})$ and hence λ^1 is bijective. Also, by Theorem 3.6 of K. Kodaira[11], λ^2 is injective.

THEOREM2.3([9,12]). *(Mayer-Vietoris). Let X be a paracompact analytic space with X_1 and X_2 open subsets of X such that $X = X_1 \cup X_2$ and $Y = X_1 \cap X_2$. Then the following sequence is exact.*

$$\cdots \to H^p(X, \mathcal{O}) \to H^p(X_1, \mathcal{O}) \oplus H^p(X_2, \mathcal{O}) \\ \to H^p(Y, \mathcal{O}) \to H^{p+1}(X, \mathcal{O}) \to \cdots$$

for the sheaf $\mathcal{O}_{X_j} =: \mathcal{O}$ and $p \geqq 0$.

3. Stein-ness of the union of Stein open subsets

Let X be a holomorphically separable complex space of finite dimension with the structure sheaf $\mathcal{O}$.

DEFINITION 3.1. The family $\mathcal{U} = \{U_j : j = 1, 2, \dots, n\}$ of subsets of X is called a finite chain cover of X if $X = \bigcup_{j=1}^{n} U_j, U_j \cap U_{j+1} \cap U_{j+2} = \emptyset$ for $j = 1, 2, \dots, n-2$, $U_{n-1} \cap U_n \cap U_1 = \emptyset$, and $U_n \cap U_1 \cap U_2 = \emptyset$.

PROPOSITION 3.2. *If X has a finite chain Stein cover $\mathcal{U} = \{U_j : j = 1, 2, \dots, n\}$ and $U_1 \cap U_n \neq \emptyset$ then the following sequences are exact:*

$$B^1(\mathcal{U}, \mathcal{O}) \to Z^1(\mathcal{U}, \mathcal{O}) \to H^1(X, \mathcal{O}),$$
$$0 \to H^0(X, \mathcal{O}) \to H^0(U_1, \mathcal{O}) \oplus H^0(U_2, \mathcal{O}) \oplus \cdots \oplus H^0(U_n, \mathcal{O})$$
$$\to H^0(U_{12}, \mathcal{O}) \oplus H^0(U_{23}, \mathcal{O}) \oplus \cdots \oplus H^0(U_{n1}, \mathcal{O})$$

where $U_{ij} = U_i \cap U_j$.

Proof. By Lemma 2.2, we have $H^1(\mathcal{U}, \mathcal{O}) = H^1(X, \mathcal{O})$. Hence we have an exact sequence

$$B^1(\mathcal{U}, \mathcal{O}) \to Z^1(\mathcal{U}, \mathcal{O}) \to H^1(X, \mathcal{O}),$$

from

$$B^1(\mathcal{U}, \mathcal{O}) \hookrightarrow Z^1(\mathcal{U}, \mathcal{O})$$

and the definition of cohomology group. For the second exect sequence we consider the functions defined by

$$\begin{aligned} H^0(X, \mathcal{O}) &\longrightarrow H^0(U_1, \mathcal{O}) \oplus H^0(U_2, \mathcal{O}) \oplus \cdots \oplus H^0(U_n, \mathcal{O}) \\ c &\longmapsto (c|_{U_1}, c|_{U_2}, \dots, c|_{U_n}) \end{aligned}$$

and

$$H^0(U_1, \mathcal{O}) \oplus H^0(U_2, \mathcal{O}) \oplus \cdots \oplus H^0(U_n, \mathcal{O})$$
$$\longrightarrow H^0(U_{12}, \mathcal{O}) \oplus H^0(U_{23}, \mathcal{O}) \oplus \cdots \oplus H^0(U_{n1}, \mathcal{O})$$
$$(c_1, c_2, \dots, c_n) \longmapsto \big((c_1 - c_2)|U_{12}, (c_2 - c_3)|U_{23}, \dots, (c_n - c_1)|U_{n1}\big).$$

DEFINITION 3.3. The family $\mathcal{U} = \{U_j : j = 1, 2, \dots, n\}$ of subsets of X is called a finite simple chain cover of X if $X = \bigcup_{j=1}^n U_j$ and $U_j \cap U_k = \emptyset$ for $| j - k | > 1$.

PROPOSITION 3.4. *Let X be a holomorphically separable complex space with the structure sheaf $\mathcal{O}$. Let Ω be an open subset of X. Suppose that X is Stein or $\Omega \Subset X$. If $dim_{\mathbb{C}} H^1(\Omega, \mathcal{O}) < \infty$ and there exists a finite simple chain Stein cover $\{\Omega_j : j = 1, 2, \ldots, n\}$ of Ω , then $\bigcup_{j=1}^{k} \Omega_j (k = 2, 3, \ldots, n)$ are Stein.*

Proof. By Theorem 3.1, we have exact sequences of cohomology groups

$$\begin{aligned}\cdots \to H^p\big(\Omega, \mathcal{O}\big) \to H^p(\cup_{j=1}^{n-1}\Omega_j, \mathcal{O}) \oplus H^p(\Omega_n, \mathcal{O}) \\ \to H^p\big(\Omega_{n-1} \cap \Omega_n, \mathcal{O}\big) \to \cdots ,\end{aligned}$$

and

$$\begin{aligned}\cdots \to H^p\big(\cup_{j=1}^{n-1}\Omega_j, \mathcal{O}\big) \to H^p\big(\cup_{j=1}^{n-2}\Omega_j, \mathcal{O}\big) \oplus H^p(\Omega_{n-1}, \mathcal{O}) \\ \to H^p(\Omega_{n-2} \cap \Omega_{n-1}, \mathcal{O}) \to \cdots .\end{aligned}$$

Since $\Omega_j (j = 1, 2, \ldots, n)$ and $\Omega_{j-1} \cap \Omega_j (j = 2, 3, \ldots, n)$ are Stein, we have $H^p(\Omega_j, \mathcal{O}) = 0 (j = 1, 2, \ldots, n)$ and $H^p(\Omega_{j-1} \cap \Omega_j, \mathcal{O}) = 0 (j = 2, 3, \ldots, n)$ for $p \geqq 1$. Thus the map

$$H^p(\Omega, \mathcal{O}) \longrightarrow H^p\big(\cup_{j=1}^{n-1}\Omega_j, \mathcal{O}\big) \quad (p \geqq 1)$$

is surjective. And we have surjections

$$H^p\big(\cup_{j=1}^{k+1}\Omega_j, \mathcal{O}\big) \longrightarrow H^p\big(\cup_{j=1}^{k}\Omega_j, \mathcal{O}\big) \quad (k = 2, 3, \ldots, n-2)$$

for $p \geqq 1$. Hence we have

$$\begin{aligned}dim_{\mathbb{C}} H^1\big(\cup_{j=1}^{2}\Omega_j, \mathcal{O}\big) \leqq dim_{\mathbb{C}} H^1\big(\cup_{j=1}^{3}\Omega_j, \mathcal{O}\big) \leqq \cdots \\ \leqq dim_{\mathbb{C}} H^1\big(\cup_{j=1}^{n}\Omega_j, \mathcal{O}\big) < \infty.\end{aligned}$$

By Theorem 1.1, we have the theorem.

4. Cousin 1–spaces

H. Cartan[4] stated that any Cousin 1-domain in $\mathbb{C}^2$ is a domain of holomorphy, and H.Behnke-K.Stein[2] proved it. It was proved that adomain Ω in $\mathbb{C}^2$ is Stein if and only if for every n-1 dimensional analytic plane $H \subset \mathbb{C}^n$ the intersection $\Omega \cap H$ is Stein and the restriction $\Gamma(\Omega, \mathcal{O}) \to \Gamma(\Omega \cap H, \mathcal{O})$ is surjective, where $\mathcal{O}$ is the structure sheaf on Ω. When n=2, this says that a domain $\Omega \subset \mathbb{C}^2$ is Stein if and only if the Cousin 1–problem is universally solvable (see [7]).

Let X be a holomorphically separable complex space of finite dimensin and let $Z(f) = \{x \in X : f(x) = 0\}$ for every holomorphic function f on X. We say that a (reduced) complex space is a Cousin 1-space if the Cousin 1–problem is universally solvable on the space. We characterize open Stein subsets in terms of the universal solvability of the Cousin 1–problem.

LEMMA 4.1. *Let X be a complexs pace and D be an infinite discrete subset of X. Then X is holomorphically convex if and only if there exists a holomorphic function $h \in \Gamma(X, \mathcal{O})$ which is unbounded on D.*

Proof. By Theorem 4.2.4 and Theorem 4.2.12 of H.Grauert -R.Remmert[7], we have the result.

From Lemma 4.1, we have the following theorem. The methods are partially based on the above [1,14].

THEOREM 4.2. *Let Ω be an open subset of X. Suppose that X is Stein or $\Omega \Subset X$. Let f be a non-constant holomorphic function in any irreducible component of X and the following conditions are satisfied:*

(1) *Every holomorphic function on $\Omega \cap Z(f)$ can be extended on Ω (i.e., $\Gamma(\Omega, \mathcal{O}) \to \Gamma(\Omega \cap Z(f), \mathcal{O})$ is surjective).*
(2) *The subspace $\Omega \cap Z(f)$ is Stein.*

Then Ω is a Stein space.

Proof. Let $X = \bigcup_{j=1}^{\infty} X_j$ where X_j are finite dimensional irreducible components of X. Let $S_j = \{f \in \Gamma(X, \mathcal{O}) : f \text{ is not constant on } X_j\}$ be an open dense subset of $\Gamma(X, \mathcal{O})$. From the Baire's category theorem $S = \bigcap_{j=1}^{\infty} S_j$ is also dense in $\Gamma(X, \mathcal{O})$ and in particular is not empty. If $f \in S$ and $x \in \Omega$, then $f - f(x)$ belongs to S' of elements of S

whose locus of zeros intersects Ω. Hence there is at least one non-constant holomorphic function f in any irreducible component of X of finite dimension such that $\Omega \cap Z(f) \neq \emptyset$. Let $\{x_n\}$ be a sequence in Ω without accumulation points in Ω. We look for a function $g \in \Gamma(\Omega, \mathcal{O})$ such that g is unbounded on $\{x_n\}$. If X is Stein and $\{x_n\}$ has no accumulation point in X, then such a $g \in \Gamma(X, \mathcal{O})$ exists from Lemma 4.1. If $\{x_n\}$ has an accumulation point in X, we may assume that $\{x_n\}$ converges to a point $x \in X$ on the boundary of Ω under the hypotheses that X is Stein or $\Omega \Subset X$. Let $f_1, f_2, \dots, f_m$ be holomorphic functions on X which separate the points of X and have only a common zero at x. Let $Z_j = \{x \in X : f_j(x) = 0\}(j = 1, 2, \dots, m)$ and

$$\mathcal{O}_{\Omega \cap Z_j} = (\mathcal{O} / f_j \mathcal{O}) |_{\Omega \cap Z_j} .$$

If $\Omega \cap Z_j = \emptyset$ for some $j = 1, 2, \dots, m$ then f_j^{-1} is unbounded on $\{x_n\}$. Otherwise, we may assume that f_1 is a non-constant holomorphic function in any irreducible component of X and $\Omega \cap Z_1$ is a (non-empty) Stein space. Set $h_j = f_j|_{\Omega \cap Z_1}$. Since $x \notin \Omega$ and $\{x\} = \bigcap_{j=1}^{m} Z_j$, $h_2, h_3, \dots, h_m$ have no common zero. Hence there exist $g_2, g_3, \dots, g_m \in \Gamma(\Omega \cap Z_1, \mathcal{O}_{\Omega \cap Z_1})$ such that $g_2 h_2 + g_3 h_3 + \cdots + g_m h_m = 1$ on $(\Omega \cap Z_1, \mathcal{O}_{\Omega \cap Z_1})$ [8, p.244]. By (1) there exist $G_2, G_3, \dots, G_m \in \Gamma(\Omega, \mathcal{O})$ such that $G_j|_{\Omega \cap Z_1} = g_j (j = 2, 3, \dots, m)$. For every $x \in X$ we consider the morphism $\sigma : \mathcal{O} \to f_1 \mathcal{O}$ defined by $\sigma(g_x) = (f_1)_x g_x$. The short exact sequence $0 \to \kappa_1 \to \mathcal{O} \to f_1 \mathcal{O}$, where $\kappa_1 = \text{Ker } \sigma$, induces the long exact sequence

$$0 \to \Gamma(\Omega, \kappa_1) \to \Gamma(\Omega, \mathcal{O}) \to \Gamma(\Omega, f_1 \mathcal{O}) \to H^1(\Omega, \kappa_1) \to \cdots .$$

Since κ_1 is a $\mathcal{O}_{\Omega \cap Z_1}$-module, we have $H^1(\Omega, \kappa_1) \cong H^1(\Omega \cap Z_1, \kappa_1 | \Omega \cap Z_1) = 0$. Therefore $\Gamma(\Omega, f_1 \mathcal{O}) = f_1 \Gamma(\Omega, \mathcal{O})$. Since $1 - G_2 f_2 - \cdots - G_m f_m \in \Gamma(\Omega, f_1 \mathcal{O}) = f_1 \Gamma(\Omega, \mathcal{O})$, there exists $G_1 \in \Gamma(\Omega, \mathcal{O})$ such that $1 = G_1 f_1 + G_2 f_2 + \cdots + G_m f_m$. Since $\{x\} = \{x \in X : f_j(x) = 0, j = 1, 2, \dots, m\}$, at least one of among $G_1, G_2, \dots, G_m$ is unbounded on $\{x_n\}$. By Lemma 4.1, Ω is holomorphically convex.

COROLLARY 4.3. *Let X be a 2-dimensional complex space and Ω be an open subset of X. Suppose that X is Stein or $\Omega \Subset X$. If for a non-constant holomorphic function f in any irreducible component of X every holomorphic function on $\Omega \cap Z(f)$ can be extended on Ω, then Ω is a Stein space.*

REMARKS. 1. Let Ω be such a subset of X as in Theorem 4.2. Let Ω a Cousin 1-space and f be a non-constant holomorphic function on X. If $\Omega \cap Z(f)$ is Stein, by Lemma of G.Berg[3], every holomorphic function on $\Omega \cap Z(f)$ can be extended on Ω. Hence we can simply induce Theorem 2 of L.M.Tovar[14].

2. When $dim_{\mathbb{C}}X = 2$, the subspace $\Omega \cap Z(f)$ is a Stein space. Hence, when $dim_{\mathbb{C}}X = 2$, Theorem 2 of L.M.Tovar[14] says that Ω is a Cousin 1-subspace if and only if Ω is a Stein space in X. This is the result of H. Cartan [4] and H.Behnke- K.Stein[2] on the complex space.

DEFINITION 4.4. We say that an open subset Ω of X is a domain of holomorphy in X if there are no open subsets Ω_1 and Ω_2 in X with the following properties:

(1) $\emptyset \neq \Omega_1 \subset \Omega_2 \cap \Omega$.
(2) Ω_2 is connected and not contained in Ω.
(3) For every holomorphic function f in Ω there is a holomorphic function F in Ω_2 such that $F = f$ in Ω_1.

THEOREM 4.5. *Let X be a Stein space and Ω be an open subset of X. Suppose that the following conditions are satisfied:*

(1) *For a non-constant holomorphic function f on X, $\Omega \cap Z(f)$ with the reduced structure is a Stein space.*
(2) *Every holomorphic function on $\Omega \cap Z(f)$ extends on Ω.*

Then Ω is a domain of holomorphy.

Proof. If not, then there are open subsets Ω_1 and Ω_2 in X with the properties of Definition 4.4. We take points $z_1 \in \Omega_1$ and $z_2 \in b\Omega \cap \Omega_2$. Let f be a non-constant holomorphic function in X such that $z_1, z_2 \in Z(f)$. From assumption (1) the subspace $\Omega \cap Z(f)$ is Stein, whence there is a holomorphic function g on $Z(f)$ which cannnot be continued over z_2. By assumption (2), g has an extension to Ω. And this extension cannot be analytically continued over $z_2 \in \Omega_2$. This contradicts the property (3) of Definition 4.4.

COROLLARY 4.6. *If Ω is a Cousin 1-space and $\Omega \cap Z(f)$ is a Stein space, then Ω is a domain of holomorphy.*

Proof. By Lemma of G.Berg[3], every holomorphic function on $\Omega \cap Z(f)$ extends on Ω. From Theorem 4.5, we have the result.

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