

**A CHANGE OF SCALE FORMULA  
FOR WIENER INTEGRALS  
ON ABSTRACT WIENER SPACES II**

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**1. Introduction**

It has long been known that Wiener measure and Wiener measurability behave badly under the change of scale transformation [3] and under translations [2]. However Cameron and Storvick [5] for a rather large class of functionals expressed the analytic Feynman integral as a limit of Wiener integrals. In doing so, they discovered a rather nice change of scale formula for Wiener integrals on classical Wiener space [6]. In [20,22,23], Yoo, Yoon and Skoug extended these results to Yeh-Wiener space and to an abstract Wiener space.

This paper continues the study of a change of scale formula for Wiener integrals on an abstract Wiener space previously given in [22]. Motivated by the work of Kallianpur and Bromley [17], we establish a change of scale formula for Wiener integrals, for a larger class than the Fresnel class studied in [22], on an abstract Wiener space. Results in [5,6,20,22,23] will then be corollaries of our results.

**2. Definitions and preliminaries**

Let  $H$  be a real separable infinite dimensional Hilbert space with inner product  $\langle \cdot, \cdot \rangle$  and norm  $\| \cdot \|$ . Let  $||| \cdot |||$  be a measurable norm on  $H$  with respect to the Gaussian cylinder set measure  $\sigma$  on  $H$ . Let  $B$  denote the completion of  $H$  with respect to  $||| \cdot |||$ . Let  $\iota$  denote the natural injection from  $H$  to  $B$ . The adjoint operator  $\iota^*$  of  $\iota$  is one-to-one and maps  $B^*$  continuously onto a dense subset of

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$H^*$ . By identifying  $H$  with  $H^*$  and  $B^*$  with  $\iota^*B^*$ , we have a triple  $B^* \subset H^* \equiv H \subset B$  and  $\langle h, x \rangle = (h, x)$  for all  $h$  in  $H$  and  $x$  in  $B$  where  $(\cdot, \cdot)$  denotes the natural dual pairing between  $B$  and  $B^*$ . By a well-known result of Gross [13],  $\sigma \circ \iota^{-1}$  has a unique countably additive extension  $m$  to the Borel  $\sigma$ -algebra  $\mathcal{B}(B)$  of  $B$ . The triple  $(H, B, m)$  is called an abstract Wiener space. For more detailed, see [12,17,18,19].

DEFINITION 2.1. Let  $F$  be a functional on  $B \times B$  such that the integral

$$(2.1) \quad J_F(\vec{\lambda}) = \int_{B \times B} F(\lambda_1^{-1/2}x_1, \lambda_2^{-1/2}x_2) d(m \times m)(x_1, x_2)$$

exists for all  $\lambda_1 > 0$  and  $\lambda_2 > 0$  where  $\vec{\lambda} = (\lambda_1, \lambda_2)$ . If there exists an analytic function  $J_F^*(\vec{z})$  on  $\Omega = \{\vec{z} = (z_1, z_2) \in \mathbb{C}^2 : \text{Re } z_k > 0 \text{ for } k = 1, 2\}$  such that  $J_F^*(\vec{\lambda}) = J_F(\vec{\lambda})$  for all  $\lambda_1 > 0$  and  $\lambda_2 > 0$ , then we define  $J_F^*(\vec{z})$  to be the analytic Wiener integral of  $F$  over  $B \times B$  with parameter  $\vec{z}$ , and for  $\vec{z} \in \Omega$  we write

$$(2.2) \quad I_a^{\vec{z}}[F(\cdot, \cdot)] = J_F^*(\vec{z}).$$

Let  $q_1$  and  $q_2$  be non-zero real numbers. If the following limit (2.3) exists, we call it the analytic Feynman integral of  $F$  over  $B \times B$  with parameter  $\vec{q} = (q_1, q_2)$  and we write

$$(2.3) \quad I_a^{\vec{q}}[F(\cdot, \cdot)] = \lim_{\vec{z} \rightarrow (-iq_1, -iq_2)} I_a^{\vec{z}}[F(\cdot, \cdot)]$$

where  $\vec{z} = (z_1, z_2)$  approaches  $(-iq_1, -iq_2)$  throughout  $\Omega$ .

Let  $\{e_n\}$  denote a complete orthonormal (C.O.N.) system in  $H$  such that the  $e_n$ 's are in  $B^*$ . For each  $h \in H$  and  $x \in B$ , we introduce a stochastic inner product  $(\cdot, \cdot)^\sim$  on  $H \times B$  defined by

$$(2.4) \quad (h, x)^\sim = \begin{cases} \lim_{n \rightarrow \infty} \sum_{k=1}^n \langle h, e_k \rangle (x, e_k), & \text{if the limit exists} \\ 0, & \text{otherwise.} \end{cases}$$

Then, for every  $h \in H$ ,  $(h, x)^\sim$  exists for  $m$ -a.e.  $x \in B$  and is a Borel measurable function on  $B$  having a Gaussian distribution with

mean 0 and variance  $\| h \|^2$ . Also if both  $h$  and  $x$  are in  $H$ , then  $(h, x)^\sim = \langle h, x \rangle$ .

Given two  $C$ -valued functions  $F$  and  $G$  on  $B \times B$ , we say that  $F = G$  s-a.e. if  $F(\alpha x_1, \beta x_2) = G(\alpha x_1, \beta x_2)$  for  $m \times m$ -a.e.  $(x_1, x_2) \in B \times B$  for all  $\alpha > 0$  and  $\beta > 0$ . For a function  $F$  on  $B \times B$ , we will denote by  $[ F ]$  the equivalence class of functions which equals  $F$  s-a.e..

DEFINITION 2.2. Let  $A_1$  and  $A_2$  be two bounded, non-negative self-adjoint operators on  $H$ . Let  $F_{A_1, A_2}$  be the space of all s-equivalence classes of functions  $F$  on  $B \times B$  which have the form

$$(2.5) \quad F(x_1, x_2) = \int_H \exp \left\{ i \left[ (A_1^{1/2} h, x_1)^\sim + (A_2^{1/2} h, x_2)^\sim \right] \right\} d\mu(h)$$

for some finite complex Borel measure  $\mu$  on  $H$ .

Let  $M(H)$  denote the space of finite complex Borel measures  $\mu$  on  $H$ .  $M(H)$  is then a Banach algebra over the complex numbers under convolution, with the norm  $\| \mu \|$  equal to the total variation of  $\mu$ . The map  $\mu \rightarrow [F]$  defined by (2.5) sets up an algebra isomorphism between  $M(H)$  and  $F_{A_1, A_2}$  if the range of  $A_1 + A_2$  is dense in  $H$ . In this case,  $F_{A_1, A_2}$  becomes a Banach algebra under the norm  $\| F \| = \| \mu \|$ .

THEOREM 2.3 ([17]). Let  $F \in F_{A_1, A_2}$  be given by (2.5). Then the analytic Wiener integral of  $F$  over  $B \times B$  with parameter  $\vec{z} = (z_1, z_2) \in \Omega$  exists and

$$(2.6) \quad I_a^{\vec{z}}[F(\cdot, \cdot)] = \int_H \exp \left\{ -(1/2) \sum_{k=1}^2 z_k^{-1} \langle A_k h, h \rangle \right\} d\mu(h).$$

Also the analytic Feynman integral of  $F$  over  $B \times B$  with parameter  $\vec{q} = (q_1, q_2)$  exists provided that  $q_1 \neq 0$  and  $q_2 \neq 0$ , and

$$(2.7) \quad I_a^{\vec{q}}[F(\cdot, \cdot)] = \int_H \exp \left\{ -(i/2) \sum_{k=1}^2 q_k^{-1} \langle A_k h, h \rangle \right\} d\mu(h).$$

In particular,

$$(2.8) \quad I_a^{(1, -1)}[F(\cdot, \cdot)] = \int_H \exp \{ -(i/2) \langle (A_1 - A_2) h, h \rangle \} d\mu(h).$$

REMARK 2.4. Let  $A$  be a bounded self-adjoint operator on  $H$ . Then we may write  $A = A^+ - A^-$  where  $A^+$  and  $A^-$  are each bounded, non-negative self-adjoint. Take  $A_1 = A^+$  and  $A_2 = A^-$  in Definition 2.2 above. For any  $F$  in  $F_{A_1, A_2}$ , (2.8) becomes  $I_a^{(1, -1)}[F(\cdot, \cdot)] = \int_H \exp\{-(i/2)\langle Ah, h \rangle\} d\mu(h)$ . Also, in this case, if  $A^+$  is the identity and  $A^- = 0$  then  $F_{A_1, A_2}$  is essentially the Fresnel class  $F(H)$  and  $F(B)$  on Hilbert space and abstract Wiener space setting respectively, and also  $I_a^{(q_1, q_2)}[F] = I_a^{q_1}[F_0]$  where  $F_0(x_1) = F(x_1, x_2)$  for all  $(x_1, x_2) \in B \times B$  and  $I_a^{q_1}[F_0]$  means the analytic Feynman integral over  $B$  [17,18]. Hence this shows that Definition 2.2 and 2.1 include the definition of Fresnel classes  $F(H)$  and  $F(B)$  and of the analytic Feynman integral over an abstract Wiener space  $B$  as a special case respectively.

### 3. Change of scale formulas

We begin this section with a key lemma for Wiener integral on an abstract Wiener space  $(H, B, m)$ .

LEMMA 3.1 ([22]). *Let  $z \in C$  with  $Re z > 0$ , let  $\{e_n\}$  be an orthonormal set in  $H$  and let  $h \in H$ . Then*

$$(3.1) \quad \int_B \exp \left\{ [(1-z)/2] \sum_{k=1}^n [(e_k, x)^\sim]^2 + i(h, x)^\sim \right\} dm(x) \\ = z^{-n/2} \exp \left\{ [(z-1)/2z] \sum_{k=1}^n [(e_k, h)]^2 - (1/2) \|h\|^2 \right\}.$$

In the following theorem, for  $F \in F_{A_1, A_2}$ , we express the analytic Feynman integral of  $F$  over  $B \times B$  as the limit of a sequence of Wiener integrals.

THEOREM 3.2. *Let  $\{e_n\}$  be a C.O.N. sequence in  $H$ . Let  $F \in F_{A_1, A_2}$  and let  $\{z_{n,k}\}$  be the sequence with  $Re z_{n,k} > 0$  such that*

$z_{n,k} \rightarrow -iq_k$  ( $q_k \neq 0$ ) for  $k = 1, 2$ . Then

$$\begin{aligned}
 (3.2) \quad & I_a^{\bar{q}}[F(\cdot, \cdot)] \\
 &= \lim_{n \rightarrow \infty} (z_{1,n} z_{2,n})^{n/2} \int_{B \times B} \\
 & \exp \left\{ \sum_{j=1}^2 \left( [(1 - z_{j,n})/2] \sum_{k=1}^n [(e_k, x_j)^\sim]^2 \right) \right\} \\
 & F(x_1, x_2) d(m \times m)(x_1, x_2).
 \end{aligned}$$

*Proof.* Since  $F \in F_{A_1, A_2}$  is given by (2.5),

$$F(x_1, x_2) = \int_H \exp \left\{ [(A_1^{1/2} h, x_1)^\sim + (A_2^{1/2} h, x_2)^\sim] \right\} d\mu(h)$$

for some  $\mu \in M(H)$ . By Fubini theorem and Lemma 3.1, we have

$$\begin{aligned}
 & \int_{B \times B} \exp \left\{ \sum_{j=1}^2 \left( [(1 - z_{j,n})/2] \sum_{k=1}^n [(e_k, x_j)^\sim]^2 \right) \right\} \\
 & F(x_1, x_2) d(m \times m)(x_1, x_2) \\
 &= (z_{1,n} z_{2,n})^{-n/2} \int_H \\
 & \exp \left\{ \sum_{j=1}^2 \left( [(z_{j,n} - 1)/2z_{j,n}] \sum_{k=1}^n [(e_k, A_j^{1/2} h)]^2 - (1/2) \langle A_j h, h \rangle \right) \right\} \\
 & d\mu(h).
 \end{aligned}$$

Next, using the bounded convergence theorem, equation (2.5) and Parseval's relation, it follows that

$$\begin{aligned}
 & \lim_{n \rightarrow \infty} (z_{1,n} z_{2,n})^{n/2} \int_{B \times B} \exp \left\{ \sum_{j=1}^2 \left( [(1 - z_{j,n})/2] \sum_{k=1}^n [(e_k, x_j)^\sim]^2 \right) \right\} \\
 & F(x_1, x_2) d(m \times m)(x_1, x_2) \\
 &= \int_H \exp \left\{ -(i/2) \sum_{k=1}^2 q_k^{-1} \langle A_k h, h \rangle \right\} d\mu(h) = I_a^{\bar{q}}[F(\cdot, \cdot)].
 \end{aligned}$$

**COROLLARY 3.3.** Let  $\{e_n\}$  be a C.O.N. sequence in  $H$ . Let  $F \in F(B)$  and let  $\{z_n\}$  be a sequence of complex numbers with  $\operatorname{Re} z_n > 0$  such that  $z_n \rightarrow -iq$  ( $q \neq 0$ ). Then the analytic Feynman integral of  $F$  over  $B$  with parameter  $q$  exists and

$$(3.3) \quad I_a^q[F] = \lim_{n \rightarrow \infty} z_n^{n/2} \int_B \exp \left\{ [(1 - z_n)/2] \sum_{k=1}^n [(e_k, x)^\sim]^2 \right\} F(x) dm(x)$$

*Proof.* Apply Theorem 3.2 after making the following choices:  $A_1 =$  the identity,  $A_2 = 0$ ,  $z_{1,n} = z_n$  and  $q_1 = q$ . With these choices and using Lemma 3.1, we can easily obtain equation (3.3).

**COROLLARY 3.4.** Under the hypothesis of Corollary 3.3, the analytic Feynman integral of  $F$  over  $B$  in (3.3) can be replaced by the sequential Feynman integral of  $F$  over  $B$ .

*Proof.* This follows from the fact that for  $F \in F(B)$  the sequential Feynman integral of  $F$  is equal to its analytic Feynman integral [18].

Our main result, namely a change of scale formula for Wiener integrals on a product abstract Wiener space now follows from Theorem 3.2 above.

**THEOREM 3.5.** Let  $\rho_1 > 0$  and  $\rho_2 > 0$  be given and let  $\{e_n\}$  be a C.O.N. sequence in  $H$ . Then for  $F \in F_{A_1, A_2}$

$$(3.4) \quad \int_{B \times B} F(\rho_1 x_1, \rho_2 x_2) d(m \times m)(x_1, x_2) \\ = \lim_{n \rightarrow \infty} (\rho_1 \rho_2)^{-n} \int_{B \times B} \exp \left\{ \sum_{j=1}^2 \left( [(\rho_j^2 - 1)/2\rho_j^2] \sum_{k=1}^n [(e_k, x_j)^\sim]^2 \right) \right\} \\ F(x_1, x_2) d(m \times m)(x_1, x_2)$$

*Proof.* A careful examination of the proof of Theorem 3.2 and using

equation (2.6) instead of (2.7) establishes the following formula:

$$\begin{aligned}
 (3.5) \quad & I_a^{(z_1, z_2)} [F(\cdot, \cdot)] \\
 &= \lim_{n \rightarrow \infty} (z_{1,n} z_{2,n})^{n/2} \int_{B \times B} \\
 & \exp \left\{ \sum_{j=1}^2 \left( [(1 - z_{j,n})/2] \sum_{k=1}^n [(e_k, x_j) \sim]^2 \right) \right\} \\
 & F(x_1, x_2) d(m \times m)(x_1, x_2)
 \end{aligned}$$

By letting  $z_k = z_{k,n} = \rho_k^{-2}$  ( $k = 1, 2$ ), (3.5) becomes equation (3.4).

The Banach algebra  $F_{A_1, A_2}$  is not closed with respect to pointwise or even uniform convergence [16,p2], and thus its uniform closure  $\overline{F}_{A_1, A_2}^u$  with respect to uniform convergence s-a.e. is a larger space than  $F_{A_1, A_2}$ . Next we show that equation (3.4) holds for  $F \in \overline{F}_{A_1, A_2}^u$ .

**THEOREM 3.6.** *Let  $\rho_1 > 0$  and  $\rho_2 > 0$  be given and let  $\{e_n\}$  be a C.O.N. set of functions in  $H$ . Then equation (3.4) holds for each  $F \in \overline{F}_{A_1, A_2}^u$ .*

*Proof.* Since  $F \in \overline{F}_{A_1, A_2}^u$ , there exists a sequence  $\{F_p\}$  from  $F_{A_1, A_2}$  such that  $F(x_1, x_2) = \lim_{p \rightarrow \infty} F_p(x_1, x_2)$  uniformly s-a.e. on  $B \times B$ . Also since each  $F_p \in F_{A_1, A_2}$ ,  $F_p(x_1, x_2)$  exists and is bounded s-a.e. on  $B \times B$  for each  $p$ . Let  $\rho_1 > 0$  and  $\rho_2 > 0$  be given. From the definition of uniform convergence s-a.e., it follows that

$$(3.6) \quad F(\rho_1 x_1, \rho_2 x_2) = \lim_{p \rightarrow \infty} F_p(\rho_1 x_1, \rho_2 x_2) \text{ uniformly a.e. on } B \times B$$

and

$$\begin{aligned}
 (3.7) \quad & \int_{B \times B} F(\rho_1 x_1, \rho_2 x_2) d(m \times m)(x_1, x_2) \\
 &= \lim_{p \rightarrow \infty} \int_{B \times B} F_p(\rho_1 x_1, \rho_2 x_2) d(m \times m)(x_1, x_2).
 \end{aligned}$$

By (3.6), there exists  $M > 0$  and a scale-invariant null set  $E$  of  $B \times B$  such that for all  $p$  and all  $(x_1, x_2) \in B \times B - E$

$$(3.8) \quad |F_p(x_1, x_2)| \leq M \quad \text{and} \quad |F(x_1, x_2)| \leq M.$$

Hence, using (3.8) and Lemma 3.1, we obtain

$$\begin{aligned} & (\rho_1 \rho_2)^{-n} \int_{B \times B} \exp \left\{ \sum_{j=1}^2 \left( [(\rho_j^2 - 1)/2\rho_j^2] \sum_{k=1}^n [(e_k, x_j) \sim]^2 \right) \right\} \\ & F_p(x_1, x_2) d(m \times m)(x_1, x_2) \\ & - (\rho_1 \rho_2)^{-n} \int_{B \times B} \exp \left\{ \sum_{j=1}^2 \left( [(\rho_j^2 - 1)/2\rho_j^2] \sum_{k=1}^n [(e_k, x_j) \sim]^2 \right) \right\} \\ & F(x_1, x_2) d(m \times m)(x_1, x_2) \quad | \\ & \leq (\rho_1 \rho_2)^{-n} \int_{B \times B} \exp \left\{ \sum_{j=1}^2 \left( [(\rho_j^2 - 1)/2\rho_j^2] \sum_{k=1}^n [(e_k, x_j) \sim]^2 \right) \right\} \\ & | F_p(x_1, x_2) - F(x_1, x_2) | d(m \times m)(x_1, x_2) \\ & \leq 2M. \end{aligned}$$

Next, using Theorem 3.2, the iterated limit theorem and the dominated convergence theorem, we obtain

$$\begin{aligned} (3.9) \quad & \int_{B \times B} F(\rho_1 x_1, \rho_2 x_2) d(m \times m)(x_1, x_2) \\ & = \lim_{p \rightarrow \infty} \int_{B \times B} F_p(\rho_1 x_1, \rho_2 x_2) d(m \times m)(x_1, x_2) \\ & = \lim_{p \rightarrow \infty} \lim_{n \rightarrow \infty} (\rho_1 \rho_2)^{-n} \int_{B \times B} \exp \left\{ \sum_{j=1}^2 \left( [(\rho_j^2 - 1)/2\rho_j^2] \sum_{k=1}^n [(e_k, x_j) \sim]^2 \right) \right\} \\ & F_p(x_1, x_2) d(m \times m)(x_1, x_2) \\ & = \lim_{n \rightarrow \infty} \lim_{p \rightarrow \infty} (\rho_1 \rho_2)^{-n} \int_{B \times B} \exp \left\{ \sum_{j=1}^2 \left( [(\rho_j^2 - 1)/2\rho_j^2] \sum_{k=1}^n [(e_k, x_j) \sim]^2 \right) \right\} \\ & F_p(x_1, x_2) d(m \times m)(x_1, x_2) \end{aligned}$$



$$= \lim_{n \rightarrow \infty} (\rho_1 \rho_2)^{-n} \int_{B \times B} \exp \left\{ \sum_{j=1}^2 \left( [(\rho_j^2 - 1)/2\rho_j^2] \sum_{k=1}^n [(e_k, x_j)^\sim]^2 \right) \right\} F(x_1, x_2) d(m \times m)(x_1, x_2).$$

The following corollary is a change of scale formula for Wiener integrals on an abstract Wiener space given in [22].

**COROLLARY 3.7.** *Let  $\rho > 0$  and let  $\{e_n\}$  be a C.O.N. sequence in  $H$ . Then for  $F \in F(B)$*

(3.10)

$$\int_B F(\rho x) dm(x) = \lim_{n \rightarrow \infty} \rho^{-n} \int_B \exp \left\{ [(\rho^2 - 1)/2\rho^2] \sum_{k=1}^n [(e_k, x)^\sim]^2 \right\} F(x) dm(x).$$

In addition, the equation (3.10) holds for  $F \in \overline{F(B)}^u$ .

*Proof.* Apply Theorem 3.5 and 3.6 after making the following choices:  $A_1 =$  the identity,  $A_2 = 0$  and  $\rho_1 = \rho$ . With these choices and Lemma 3.1, we can easily obtain our corollary.

Finally we end this section by showing that the class of functions for which Theorem 3.5 (Theorem 3.6) and Corollary 3.7 hold is more extensive than  $\overline{F}_{A_1, A_2}^u$  and  $\overline{F(B)}^u$  respectively.

**REMARK 3.8.** Let  $\{e_n\}$  be a C.O.N. sequence in  $H$  and let  $F$  be the function of the form  $F(x) = \exp\{\alpha(h, x)^\sim\}$  for non-zero  $h \in H$  where  $\alpha$  is a real or complex number. Then the left side of the equation (3.10) becomes

$$\begin{aligned} & \int_B F(\rho x) dm(x) \\ &= [1/\sqrt{2\pi} \|h\|] \int_R \exp\{\alpha \rho y - y^2/2 \|h\|^2\} dy \\ &= \exp\{(\alpha \rho \|h\|)^2/2\}. \end{aligned}$$

To evaluate the integral on the right side of (3.10), we apply some technique in the proof of Lemma 3.1 so that

$$\begin{aligned} & \int_B \exp \left\{ [(\rho^2 - 1)/2\rho^2] \sum_{k=1}^n [(e_k, x)^\sim]^2 \right\} F(x) dm(x) \\ &= \int_B \exp \left\{ [(\rho^2 - 1)/2\rho^2] \sum_{k=1}^n [(e_k, x)^\sim]^2 + \alpha(h, x)^\sim \right\} dm(x) \\ &= \rho^n \exp \left\{ [(\alpha\rho)^2/2] \sum_{k=1}^n [(e_k, h)]^2 + (\alpha^2/2) \left( \|h\|^2 - \sum_{k=1}^n [(e_k, h)]^2 \right) \right\}. \end{aligned}$$

Using the Parseval's relation, we obtain

$$\begin{aligned} & \lim_{n \rightarrow \infty} \rho^{-n} \int_B \exp \left\{ [(\rho^2 - 1)/2\rho^2] \sum_{k=1}^n [(e_k, x)^\sim]^2 \right\} F(x) dm(x) \\ &= \exp \{ (\alpha\rho \|h\|)^2/2 \}. \end{aligned}$$

Thus we established that equation (3.10) is valid for all complex number  $\alpha$ . In particular, if  $\alpha$  is pure imaginary, then  $F$  belongs to  $F(B)$ . On the other hand, if  $\operatorname{Re} \alpha \neq 0$ , then  $F$  is unbounded so  $F \notin F(B)$  and also  $F \notin \overline{F(B)}^u$ .

#### 4. Examples

Let  $H_N$  be the set of all functions  $f : [0, 1]^N \rightarrow R$  for which there exists  $v$  in  $L_2([0, 1]^N)$  such that

$$(4.1) \quad f(t_1, \dots, t_N) = \int_0^{t_N} \cdots \int_0^{t_1} v(s_1, \dots, s_N) ds_1 \cdots ds_N$$

for all  $(t_1, \dots, t_N) \in [0, 1]^N$ . The inner product on  $H_N$  is defined by

$$(4.2) \quad \langle f, g \rangle_{H_N} = \int_{[0, 1]^N} [Df(t_1, \dots, t_N)] [Dg(t_1, \dots, t_N)] dt_1 \cdots dt_N$$

where  $D(\cdot) = \partial^N(\cdot)/\partial t_1 \cdots \partial t_N$ . Then  $H_N$  is a separable Hilbert space. Let  $B_N = C_N([0, 1]^N)$  denote the space of all  $R$ -valued continuous functions  $x$  on  $[0, 1]^N$  such that  $x(t_1, \cdots, t_N) = 0$  whenever at least one of  $t_1, \cdots, t_N$  equals zero. Let  $m_N$  denote  $N$ -parameter Wiener measure on  $(B_N, \mathcal{B}(B_N))$ . Then  $(H_N, B_N, m_N)$  is an example of abstract Wiener space.

Let  $f$  and  $g$  be elements in  $H_N$  such that

$$f(t_1, \cdots, t_N) = \int_0^{t_N} \cdots \int_0^{t_1} v(s_1, \cdots, s_N) ds_1 \cdots ds_N$$

and

$$g(t_1, \cdots, t_N) = \int_0^{t_N} \cdots \int_0^{t_1} u(s_1, \cdots, s_N) ds_1 \cdots ds_N$$

for some  $v$  and  $u$  in  $L_2([0, 1]^N)$ . Then

$$(4.3) \quad \langle f, g \rangle_{H_N} = \langle v, u \rangle_{L_2}.$$

Also if  $\{e_n\}$  is a C.O.N. sequence in  $H_N$ , we can express the  $e_n$ 's by

$$e_n(t_1, \cdots, t_N) = \int_0^{t_N} \cdots \int_0^{t_1} \phi_n(s_1, \cdots, s_N) ds_1 \cdots ds_N$$

for some  $\phi_n$  in  $L_2([0, 1]^N)$ . From (4.3) it follows that  $\{\phi_n\}$  is also a C.O.N. sequence in  $L_2([0, 1]^N)$ . And hence  $(e_n, x)^\sim$  equals the Paley-Wiener-Zygmund stochastic integral

$$(e_n, x)^\sim = \int_{[0, 1]^N} \phi_n(s_1, \cdots, s_N) \widetilde{dx}(s_1, \cdots, s_N)$$

for s-a.e.  $x \in B_N$ .

Let  $\nu$  be a positive integer. Let  $H_N^\nu$  denote the  $\nu$ -copies  $X_1^\nu H_N$  of  $H_N$  with inner product

$$\langle (f_1, \cdots, f_\nu), (g_1, \cdots, g_\nu) \rangle = \sum_{k=1}^{\nu} \langle f_k, g_k \rangle_{H_N}.$$

Then  $(H_N^\nu, B_N^\nu, m_N^\nu)$  is the classical  $N$ -parameter Wiener measure space in  $\nu$ -dimension.

Let  $S_N(\nu)$  be the space of functions  $F$  on  $B_N^\nu$  of the form  
 (4.4)

$$F(\vec{x}) = \int_{L_2^\nu([0,1]^N)} \exp \left\{ i \sum_{k=1}^{\nu} \int_{[0,1]^N} v_k(t_1, \dots, t_N) \widetilde{dx}_k(t_1, \dots, t_N) \right\} d\mu(\vec{v})$$

for s-a.e.  $\vec{x}$  in  $B_N^\nu$  where  $\mu \in M(L_2^\nu([0,1]^N))$  in [21].

COROLLARY 4.1. Let  $\{\phi_n\}$  be a C.O.N. set of functions on  $[0,1]^N$ . Then for each  $\rho > 0$  and each  $F \in S_N(\nu)$

(4.5)

$$\int_{B_N^\nu} F(\rho\vec{x}) dm_N^\nu(\vec{x}) = \lim_{n \rightarrow \infty} \rho^{-\nu n} \int_{B_N^\nu} \exp \left\{ [(\rho^2 - 1)/2\rho^2] \sum_{j=1}^{\nu} \sum_{k=1}^n \left[ \int_{[0,1]^N} \phi_k(t_1, \dots, t_N) \widetilde{dx}_j(t_1, \dots, t_N) \right]^2 \right\} F(\vec{x}) dm_N^\nu(\vec{x}).$$

*Proof.* Let  $e_k(t_1, \dots, t_N) = \int_0^{t_1} \dots \int_0^{t_1} \phi_k(s_1, \dots, s_N) ds_1 \dots ds_N$  and use Corollary 3.7 and the fact that  $S_N(\nu)$  and  $F(B_N^\nu)$  are isometrically isomorphic [14,17].

COROLLARY 4.2. Equation (4.5) holds for all  $F \in \overline{S}_N^u(\nu)$ .

COROLLARY 4.3. Theorem 2 and Theorem 4 in [6] follow from the case  $N = 1$  of Corollary 4.1 and Corollary 4.2 above respectively.

COROLLARY 4.4. Theorem 4.2 in [20] and Theorem 2.4 in [23] follow from the case  $N = 2$  and  $\nu = 1$  of Corollary 4.1 and Corollary 4.2 above respectively.

Ahn, Johnson and Skoug [1] introduced a general theorem insuring that many functions of interest in Feynman integration theory and quantum mechanics are in  $F(B)$  for various abstract Wiener space  $(H, B, m)$ . But this general theorem also can be extended for a larger class  $F_{A_1, A_2}$  than  $F(B)$ .

**COROLLARY 4.5.** *All of the functions discussed in Corollary 1 - 11 in [1] satisfy the change of scale formula (3.10) where  $\{e_n\}$  is a C.O.N. set of functions in the corresponding Hilbert space  $H$ .*

**COROLLARY 4.6.** *All of the functions considered in [4,8-11,16-18] satisfy the change of scale formula (3.10) where  $\{e_n\}$  is a C.O.N. set of functions in the appropriate Hilbert space  $H$ .*

The following corollaries show that the class of functionals for which the above corollaries hold is more extensive than  $\overline{S}_N^u(\nu)$ .

**COROLLARY 4.7 (EXAMPLE IN [6]).** *Let  $F(x) = \exp\{\alpha \int_0^\pi x(s) \cos s ds\}$  on  $B_1 = C_1[0, \pi]$  where  $\alpha$  is a real or complex number. If  $\text{Re } \alpha \neq 0$  then  $F \notin \overline{S}_1^u(1)$ , but it satisfies the change of scale formula (4.5) in the case of  $N = 1$  and  $\nu = 1$ .*

*Proof.* Let  $\{\phi_n(s) = (2/\pi)^{1/2} \sin ns\}$  be a C.O.N. set on  $[0, \pi]$ . By letting  $e_n(s) = \int_0^s \phi_n(t) dt$  for each  $n$ ,  $\{e_n(s)\}$  is also a C.O.N. set in  $H_1$ . Use Remark 3.8 with  $h = \cos s$  and the fact that  $S_1(1)$  is isometrically isomorphic to  $F(B_1)$ .

**COROLLARY 4.8 (EXAMPLE IN [23]).** *Let  $F(x) = \exp\{\alpha \int_0^\pi \int_0^\pi x(s, t) \cos s \cos t ds dt\}$  on  $B_2 = C_2([0, \pi]^2)$  where  $\alpha$  is a real or complex number. If  $\text{Re } \alpha \neq 0$  then  $F \notin \overline{S}_2^u(1)$ , but it satisfies the change of scale formula (4.5) in the case of  $N = 2$  and  $\nu = 1$ .*

*Proof.* Let  $e_{j,k}(s, t) = \int_0^s \int_0^t \phi_{j,k}(u, v) dv du$  for each  $j$  and  $k$  where  $\{\phi_{j,k}(s, t) = (2/\pi) \sin js \sin kt \text{ for } j, k = 1, 2, \dots\}$  is a C.O.N. set on  $[0, \pi] \times [0, \pi]$ . Use Remark 3.8 with  $h = \cos s \cos t$  and the fact that  $S_2(1)$  is isometrically isomorphic to  $F(B_2)$ .

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