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A CHARACTERIZATION OF RELATIVE ARTINIAN MODULES

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Modules noetherian and artinian relative to torsion theory were first studied by Năstăsescu and Nita.

After that many people characterized artinian and noetherian modules relative to torsion theory, for example Benander, Garcia, Golan, Lau, Leu, Manocha, Nastasesucu, Teply and recently Gomez and Torrecillas ([3]). For the history of this, we can refer to the Golan's book ([2]).

In this paper we want to give a new characterization of relative artinian modules with respect to a hereditary torsion theory; this result (Proposition 4) seems to be a dual statement of the following ([2], 20.1. (6)):

(*) An R-module M is τ-noetherian if and only if the lattice of τ-closed τ-finitely generated submodules of M satisfies the ascending chain condition.

Throughout this paper, R will denote an associative ring with identity and R-Mod will denote the category of all unitary left R-modules.

Notation and terminology concerning (hereditary) torsion theories on R-Mod will follow [2]. In particular, if τ is a torsion theory on R-Mod, then a left submodule N of M is said to be τ-closed submodule of M if and only if M/N is τ-torsionfree.

If M is a left R-module then we denote $\tau(M)$ the unique largest submodule of M which is τ-torsion. And we denote $E(M)$ the injective hull of a left R-module M.

We call a left R-module M τ-artinian (τ-noetherian, resp.) if M has descending chain condition (ascending chain condition, resp.) on τ-closed submodules of M.

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DEFINITION. *If τ is a torsion theory on R-Mod then a left R-module M is said to be τ-cofinitely generated if and only if for any set $\{M_i | i \in I\}$ of τ-closed submodules of M satisfying $\cap_{i\in I} M_i = \tau(M)$ there exists a finite subset F of I such that $\cap_{j\in F} M_j = \tau(M)$.*

PROPOSITION 1 (GARCIA-GOMEZ [1]). *If τ is a torsion theory on R-Mod, then the class of all τ-cofinitely generated left R-modules is closed under extensions.*

We will use the following well-known property frequently without explict mention. For the completeness of this paper, we give the proof in here.

PROPOSITION 2. *Let M be a left R-module, then we have the following M is τ-artinian, if and only if any τ-torsionfree homomorphic image of M is τ-cofinitely generated.*

Proof. Let M/L be a homomorphic image of M and let $\{M_i/L| i \in I\}$ be a set of τ-closed submodules of M/L satisfying $\cap_{i\in I} M_i/L = \tau(M/L)$. For each finite subset F of I let $M_F = \cap_{j\in F} M_j$ is a τ-closed submodule of M since each M_i is τ-closed submodule of M. Consider $\{M_F | F$ is a finite subset of $I\}$ which has a minimal element, say M_J. And so $M_J/L = \tau(M/L)$. Thus M/L is τ-cofinitely generated.

Converse statement is clear.

We define an R-module M is *cocyclic* if it is an essential extension of a simple R-module.

LEMMA 3. *Every non-zero τ-torsionfree R-module M has a non-zero τ-torsionfree cocyclic quotient module.*

Proof. Choose a non-zero element m in M and L a maximal τ-closed submodule of Rm.

Then there exists a non-zero homomorphism $f : Rm \longrightarrow E\ (Rm/L)$, and by the τ-torsionfree injectivity of $E(Rm/L)$, there exists a non-zero homomorphism $g : M \longrightarrow E(Rm/L)$.

Thus $M/\mathrm{ker} g$ is non-zero and can be embedded in τ-torsionfree left R-module $E(Rm/L)$.

Using above properties, we give a new characterization on τ-artinian modules this seems a dual statement of $(*)$.

PROPOSITION 4. *Let τ be a torsion theory on R-Mod. A left R-module M is τ-artinian if and only if it satisfies the following condition* :

(**) Every non-empty family of τ-cofinitely generated τ-torsionfree quotient modules of M has a maximal member.

Proof. If M is τ-artinian, the condition (**) holds evidently. Conversely assume that the condition (**) holds but M is not τ-artinian.

Then there exists a τ-torsionfree quotient module M/N of M that is not τ-cofinitely generated.

In particular we can say that M/N is non-zero. Let $\{M/N_i | i \in I\}$ be the family of non-zero τ-torsionfree τ-cofinitely generated quotient modules of M/N. By Lemma 3, I is non-empty set. Let M/N_0 be a maximal member of this family. Then N_0/N is not τ-cofinitely generated, for otherwise the exact sequence

$$0 \longrightarrow N_0/N \longrightarrow M/N \longrightarrow M/N_0 \longrightarrow 0$$

will imply that M/N is τ-cofinitely generated, by the Proposition 1.

In particular by the Lemma 3 the non-zero τ-torsionfree R-module N_0/N has a non-zero τ-torsionfree cocyclic quotient module, say N_0/L, and N is a τ-closed submodule of L. But the exact sequence

$$0 \longrightarrow N_0/L \longrightarrow M/L \longrightarrow M/N_0 \longrightarrow 0$$

implies that M/L is (τ-torsionfree) τ-cofinitely generated by the Proposition 1. And this contradicts the maximality of M/N_0. Thus we have that M is τ-artinian.

Using our characterization, we try to prove the following well-known result directly.

PROPOSITION 5. *Let τ be a torsion theory on R-Mod. If M is τ-artinian left R-module then every non-zero τ-torsionfree homomorphic image of M has a τ-cocritical submodule.*

Proof. Let M/N be a non-zero τ-torsionfree homomorphic image of M. Consider a family $\mathcal{M} = \{M/N_i|\ i \in I,\ N_i \geq N\}$ each M/N_i is τ-torsionfree, τ-cofinitely generated. By the Proposition 4, there exists a maximal element, say M/K. Then K/N is τ-cocritical submodule of

M/N. Suppose that N_1/N is non - τ - dense in K/N. Let N_2/N be the τ-closure of N_1/N in K/N. Then N_2 is τ-closed submodule of K containing N_1 (also N).

By the maximality M/K in the class $\mathcal{M}$ and M/K is τ-cofinitely generated, we have that $N_2 = K$. Thus we have the desired result.

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