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A FOCAL MYERS–GALLOWAY THEOREM ON SPACE–TIMES

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1. Introduction

Let M^n be a Riemannian manifold and γ a geodesic joining two points of M^n. Recall that Myers[13] actually showed that if along γ the Ricci curvature, Ric, satisfies

$$Ric(T,T) \geq a > 0$$

and the length of γ exceeds $\pi\sqrt{n-1}/\sqrt{a}$ where T is the unit tangent to γ, then γ is not minimal.

Moreover, there have been several applications of Myers method to general relativity. T. Frankel[7] has used Myers theorem to obtain a bound on the size of a fluid mass in stationary space-time universe. In [8], G. Galloway made use of Frankel's method to obtain a closure theorem which has as its conclusion the "finiteness" of the "spatial part" of a space-time obeying certain cosmological assumptions for cosmological models more general than the classical Friedmann models. To prove the closure theorem he generalized the Myers theorem on a Riemannian manifold. S. Markvorsen[12] obtained another extension of the Myers theorem.

On the other hand, J. K. Beem and P. E. Ehrlich[1,2] proved that if (M,g) is a globally hyperbolic space-time with all Ricci curvature positive and bounded away from zero, then (M,g) has finite timelike diameter.

In this paper, we used the generalized Myers theorem on Riemannian manifolds given by G. Galloway[8] to extend the Lorentzian version of Myers theorem given by J. K. Beem and P. E. Ehrlich. Moreover,

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we compute the upper bound of Lorentzian arc lengths of all future-directed nonspacelike curves starting from a compact spacelike submanifold K to any chronologically related point q of M for the suitable curvature tensor and second fundamental tensor conditions.

2. Preliminaries

Let (M, g) be an arbitrary space-time. Given $p, q \in M$, $p \leqq q$ means that $p = q$ or there is a piecewise smooth future directed nonspacelike curve from p to q. Let $\Omega_{p,q}$ denote the path space of all piecewise smooth future directed nonspacelike curves $\gamma : [0,1] \to M$ with $\gamma(0) = p$ and $\gamma(1) = q$. The *Lorentzian arc length* $L : \Omega_{p,q} \to \mathbf{R}$ is then defined as follows. Given a piecewise smooth curve $\gamma \in \Omega_{p,q}$, choose a partition $0 = t_0 < t_1 < t_2 < \ldots < t_n = 1$ such that $\gamma|(t_i, t_{i+1})$ is smooth for each $i = 0, 1, 2, \ldots, n-1$, and set

$$L(\gamma) = \sum_{i=0}^{n-1} \int_{t=t_i}^{t_{i+1}} \sqrt{-g(\gamma'(t), \gamma'(t))}dt.$$

Moreover, the *Lorentzian distance* $d : M \times M \to \mathbf{R} \cup \{\infty\}$ of (M, g) is defined as follows. Given $p \in M$, if $q \in J^+(p) = \{q \in M | p \leq q\}$, set $d(p,q) = \sup\{L(\gamma) : \gamma \in \Omega_{p,q}\}$, and zero otherwise. Now, we define the *timelike diameter*, $diam(M, g)$, of the space-time by

$$diam(M, g) = \sup\{d(p,q) | p, q \in M\}.$$

A space-time (M, g) is *strongly causal* if (M, g) does not contain any point p of M such that there are future-directed nonspacelike curves leaving arbitrarily small neighborhood of p and then returning. Moreover, a strong causal space-time (M, g) is said to be *globally hyperbolic* if $J^+(p) \cap J^-(q)$ is compact for all $p, q \in M$ where $J^-(p) = \{q \in M | q \leq p\}$. It should be noted that global hyperbolicity does not imply any of geodesic completeness. This may be seen by fixing points p and q in the Minkowski space L^2 with $p << q$ ($p << q$ means that there is a future-directed piecewise smooth timelike curve from p to q). Now set $M = I^+(p) \cap I^-(q)$ (here $I^+(p) = \{q \in L^2 | p << q\}$ and $I^-(q) = \{p \in L^2 | p << q\}$) equipped with the induced Lorentzian metric as an open subset of L^2. Clearly, M is totally geodesic and globally hyperbolic. Therefore, a geodesic joining any pair of causally related points in M is a geodesic segment in L^2 defined on a finite time interval.

However, the global hyperbolicity still guarantees the existence of a maximal geodesic $\gamma \in \Omega_{p,q}$, i.e., a future directed nonspacelike geodesic γ from p to q with $L(\gamma) = d(p,q)$. c.f. [2, Theorem 5.1]. This fact makes Theorem 4.2 available for the globally hyperbolic space-times.

With respect to the conjugate points it is well known that a timelike geodesic is not maximal beyond the first conjugate point (c.f. [2. p.228]).

Let $\gamma : [0,b] \longrightarrow (M,g)$ be a unit timelike geodesic segment. One considers an $\mathbf{R}$-vector space $V^{\perp}(\gamma)$ of continuous piecewise smooth vector fields Y along γ perpendicular to γ' and let $V_0^{\perp}(\gamma) = \{Y \in V^{\perp}(\gamma) | Y(0) = Y(b) = 0\}$. Then, from the second variation formula of γ, the *Lorentzian index form* $I : V^{\perp}(\gamma) \times V^{\perp}(\gamma) \longrightarrow \mathbf{R}$ is given by, for $X, Y \in V^{\perp}(\gamma)$

$$I(X,Y) = -\int_0^b \left[g(X',Y') - g\big(R(X,\gamma')\gamma',Y\big)\right] dt$$

where R is the curvature tensor with respect to the Levi-Civita connection ∇ on (M,g). Moreover, $t_1, t_2 \in [0,b]$ with $t_1 \neq t_2$ are *conjugate* with respect to the timelike geodesic γ if there is a nontrivial Jacobi field J (i.e., $J'' + R(J,\gamma')\gamma' = 0$) along γ with $J(t_1) = J(t_2) = 0$. Then we have the following maximality property of Jacobi fields with respect to the index form, cf. [1,2].

PROPOSITION 2.1. *Let $\gamma : [0,b] \longrightarrow (M,g)$ be a unit speed timelike geodesic with no conjugate points and let $J \in V^{\perp}(\gamma)$ be any Jacobi field. Then, for any $Y \in V^{\perp}(\gamma)$ with $Y \neq J$ and $Y(0) = J(0)$, $Y(b) = J(b)$, we have $I(J,J) > I(Y,Y)$.*

COROLLARY 2.2. *Let $\gamma : [0,b] \to M$ have no conjugate points. Then the index form I is negative definite on $V_0^{\perp}(\gamma) \times V_0^{\perp}(\gamma)$.*

In [1,2], J. K. Beem and P. E. Ehrlich used Corollary 2.2 to prove the Lorentzian version of Myers theorem for complete Riemannian manifolds given in [3,9] as follows.

THEOREM 2.3. *Let (M, g) be a globally hyperbolic space-time of dimension $n \geq 2$ satisfying*

$$Ric(\gamma', \gamma') \geq (n-1)k > 0$$

for any unit timelike geodesic γ. Then

$$diam(M, g) \leq \pi/\sqrt{k}.$$

In fact, if $(n-1)k = a$, we may check that this theorem reduces to Myers result on complete Riemannian manifolds.

THEOREM 2.4(MYERS-GALLOWAY). *Let M^n be a complete Riemannian manifold. Suppose there exist constants $a > 0$ and $c \geq 0$ such that for every pair of points in M^n and unit minimal geodesic γ joining those points, the Ricci curvature satisfies*

$$Ric(\gamma', \gamma') \geq a + \frac{df}{ds}$$

along γ, where f is some function of arc lengths satisfying $|f(s)| \leq c$ along γ. Then M^n is compact and

$$diam(M^n) \leq \frac{\pi}{a}\left(c + \sqrt{c^2 + a(n-1)}\right).$$

In above Myers-Galloway theorem on a Riemannian manifold we may find a differentiable function f of arc length s such that $|f(s)| \leq c$ for some $c > 0$. Such a function may be applied to prove a closure theorem of a more generalized model $(M^4, <, >)$ than the Friedmann model of general relativity. More in detail, let s be the arc length of a geodesic γ with unit tangent X in the "spatial part" V^3 of M^4 and let U be a smooth unit future-directed timelike vector field on M^4 orthogonal to V^3. Extend X along the flow lines through γ by making it invariant under the flow generated by U. Then we may set $f(s) = < X, \nabla_U U > (s)$ (c.f. [8]).

Now, we may prove the Lorentzian version of Myers-Galloway theorem as follows.

PROPOSITION 2.5. *Let (M, g) be an arbitrary space-time of dimension $n \geq 2$ and, let $\gamma : [0, b] \longrightarrow (M, g)$ be any unit timelike geodesic joining any pair of causally related points of M with length L. Suppose that*

$$Ric(\gamma', \gamma') \geq a + \frac{df}{ds}$$

where $a > 0$, f is a differentiable function of arc length s with $|f(s)| \leq c$ along γ, and $L > \frac{\pi}{a}\left(c + \sqrt{c^2 + a(n-1)}\right)$. Then γ can not be maximal.

The proof is similar to Theorem 2.3 (c.f. [2]). Note that if $f = c = 0$ then Proposition 2.5 reduces to Theorem 2.3 (Myers theorem on space-times). Moreover, in this Proposition 2.5 the Ricci curvature does not require positiveness along γ. Similarly, we have the Lorentzian analogue of Myers-Galloway diameter theorem for complete Riemannian manifolds.

THEOREM 2.6. *Let (M, g) be a globally hyperbolic space-time of dimension $n \geq 2$ and suppose there exist constants $a > 0$ and $c \geq 0$ such that for every pair of causally related points in M and any unit maximal timelike geodesic γ joining those points,*

$$Ric(\gamma', \gamma') \geq a + \frac{df}{ds}$$

where f is some function of arc lengths satisfying $|f(s)| \leq c$ along γ. Then

$$diam(M, g) \leq \frac{\pi}{a}\left(c + \sqrt{c^2 + a(n-1)}\right).$$

3. Existence of Maximal Geodesics orthogonal to the Spacelike Submanifolds

Let K be a spacelike submanifold of dimension $k \geq 0$ and let for $q \in M$, $K << q$ if there exists $p \in K$ such that $p << q$. $K \leq q$ if there exists $p \in K$ with $p \leq q$. And let $I^+(K) = \{q \in M | K << q\}$ chronological future of K, $I^-(K) = \{q \in M | q << K\}$ chronological past of K, $J^+(K) = \{q \in M | K \leq q\}$ causal future of K, $J^-(K) = \{q \in M | q \leq K\}$ causal past of K. Clearly, $I^+(K) = \cup_{p \in K} I^+(p)$.

Now, let $\Omega_{K,q}$ be the path space of all piecewise smooth future directed nonspacelike curves $\gamma : [0,b] \to (M,g)$ with $\gamma(0) \in K$ and $\gamma(b) = q$. The *Lorentzian arc length* $L : \Omega_{K,q} \longrightarrow \mathbf{R}$ is defined as in Section 2.

Now we define the *Lorentzian distance* from K to q by

$$d(K,q) = \begin{cases} 0, & \text{if } q \notin J^+(K); \\ \sup\{L(\gamma) | \gamma \in \Omega_{K,q}\}, & \text{if } q \in J^+(K). \end{cases}$$

Clearly, $d(K,q) > 0$ iff $q \in I^+(K)$. $q \in J^+(K) - I^+(K)$ implies that $d(K,q) = 0$. But the converse does not hold, since $d(K,q) = 0$ for $q \notin J^+(K)$.

Given a timelike curve γ from K to q, we have a variation α of $\gamma(t)$ and define the variation vector filed V of α along γ by

$V(t) = \frac{\partial}{\partial s}\alpha(t,s)|_{s=0}$, $V(b) = 0$, $V(0) \in T_{\gamma(0)}K$.

Then we have some facts:

if $\gamma : [0,b] \to (M,g)$ is a unit speed timelike geodesic from K to q, then $L'(0) = g(V(0), \gamma'(0))$. Thus, γ is extremal iff γ is orthogonal at $\gamma(0)$ to K.

Moreover, if $\gamma : [0,b] \to (M,g)$ is a unit timelike geodesic which is orthogonal at $\gamma(0)$ to the spacelike submanifold K and if V is a piecewise smooth vector field along γ orthogonal to γ', then we have

$$L''(0) = g\big(S_{\gamma'(0)}V(0), V(0)\big) + I(V,V)$$

where $I(V,V) = -\int_0^b [g(V',V') - g(R(V,\gamma')\gamma', V)]\, dt$ and, $S_{\gamma'(0)}$ is the second fundamental tensor given by $S_{\gamma'}x = -(\nabla_x \gamma'(0))^T$ for $x \in T_pK$ where T means "tangential part".

Hence we may define the *Lorentzian submanifold index form*

$$I_{(b,K)} : V^{\perp}(\gamma,K) \times V^{\perp}(\gamma,K) \to \mathbf{R}$$

on $V^{\perp}(\gamma,K)$ the vector space of piecewise smooth vector fields Y with $Y \perp \gamma', Y(0) \in T_pK$ as follows; for $X, Y \in V^{\perp}(\gamma,K)$,

$$I_{(b,K)}(X,Y) = g\left(S_{\gamma'(0)}X(0), Y(0)\right) + I(X,Y)$$

where I is the index form on $V^{\perp}(\gamma)$.

Now a smooth vector field $J \in V^{\perp}(\gamma, K)$ is called a *K-Jacobi field along γ* if J satisfies

(1) $J'(0) + S_{\gamma'(0)}J(0) \in (T_pK)^{\perp}$,

(2) $J'' + R(J, \gamma')\gamma' = 0$.

Hence we may define a *K-focal point* $\gamma(t_0), t_0 \in (0, b]$ if there is a nontrivial K-Jacobi field with $J(t_0) = 0$. Now, we may prove the maximality theorem of K-Jacobi fields among piecewise smooth vector fields in $V^{\perp}(\gamma, K)$ (c.f. [6]).

THEOREM 3.1. *(Maximaity of K-Jacobi fields) Let $\gamma : [0, b] \to M$ be a timelike geodesic orthogonal at $\gamma(0)$ to the spacelike submanifold K with no K-focal points and let $X \in V^{\perp}(\gamma, K)$. If $J \in V^{\perp}(\gamma, K)$ is a K-Jacobi field along γ with $J(b) = X(b)$, then*

$$I_{(b,K)}(X, X) \leq I_{(b,K)}(J, J),$$

and equality holds if and only if $X = J$.

Let $V_0^{\perp}(\gamma, K)$be the subspace of $V^{\perp}(\gamma, K)$ with $Y(b) = 0$.

COROLLARY 3.2. *If such a γ in Theorem 3.1 has no K-focal points. Then the index form $I_{(b,K)}$ is negative definite on $V_0^{\perp}(\gamma, K) \times V_0^{\perp}(\gamma, K)$.*

Recently in [4,5], P. E. Ehrlich and S. B. Kim used the maximality theorem of K-Jacobi fields to extend the Morse index theorem and the Rauch comparison theorem to the K-focal sense for nonspacelike geodesics.

Using the index form $I_{(b,K)}$ it is well known that a timelike geodesic orthogonal to a spacelike hypersurfaces K fails to maximize arc length after the first K-focal point (c.f. [2,10]). Moreover, even if K is a spacelike submanifolds of codimension arbitrary, we can easily show the proof of the following proposition as in [2] by using the fact that given a K-Jacobi field J_1, there is a vector $n \in (T_pK)^{\perp}$ such that $J_1'(0) = -S_{c'(0)}J_1(0) + n$ with $g(n, J_1(0)) = 0$.

PROPOSITION 3.3. *Let $\gamma : [0, b] \to M$ be a unit speed timelike geodesic segment orthogonal to a spacelike submanifold K at $\gamma(0) = p \in K$. If there exists $t_0 \in (0, b)$ such that $\gamma(t_0)$ is a K-focal point along γ, then there exists a variation vector field $Z \in V^{\perp}(\gamma, K)$ such* *that $I_{(b,K)}(Z,Z) > 0$, i.e., there exists a timelike curve from K to q longer than γ.*

Let K be a spacelike submanifold of a space-time (M,g). If (M,g) is a globally hyperbolic space-time, we know that there is a future directed maximal nonspacelike geodesic between any causally related two points. However, we can not guarantee the existence of the future directed maximal nonspacelike geodesic from K to a point in M (even if K is closed). It may be seen by fixing points $p = (0,-1)$ and $q = (2,3)$ in Minkowski space L^2, and by setting $K = \{(x,y)|-1 < x < 1, y = 0\}$. Then $M = I^+(p) \cap I^-(q)$ is globally hyperbolic and K is closed in M. Thus, we can not find any future directed maximal timelike geodesic from K to the point $r = (2,2)$. Moreover, we need to find such a geodesic γ orthogonal at $\gamma(0)$ to K as follows (c.f. [14]).

If M is globally hyperbolic and if $J^-(q) \cap K$ is compact, then the function $x \to d(x,q)$ is continuous on the compact set $J^-(q) \cap K$. Hence, it has a maximum at $p \in J^-(q) \cap K$. Thus, $d(K,q) = d(p,q)$. Therefore, there is a geodesic γ from p to q of length $d(K,q) = d(p,q)$. We may assume that $q \notin K$ and $p << q$. From the first variation formula, it is normal.

PROPOSITION 3.4. *Let (M,g) be a globally hyperbolic space-time and let K be a spacelike submanifold of (M,g). Then for any $q \in I^+(K)$ with $J^-(q) \cap K$ compact, there is a future directed maximal timelike geodesic γ perpendicular at $\gamma(0)$ to K in $\Omega_{K,q}$.*

4. The Main Results

Now, we generalize Proposition 2.5 to the K-focal sense.

THEOREM 4.1. *Let (M,g) be a space-time of dimension ≥ 2 and γ any unit speed timelike geodesic with length L in $\Omega_{K,q}$ perpendicular at $\gamma(0)$ to the spacelike submanifold K of dimension $k \geq 0$ for any point $q \in M$. Suppose*

$$g\big(R(u,\gamma'(t))\gamma'(t),u\big) \geq \frac{1}{n-1}\Big(a + \frac{df}{dt}\Big)$$

for all $u \in (\gamma'(t))^\perp$ with $g(u,u) = 1$ along γ, and suppose

$$g(S_{\gamma'(0)}w, w) \geq \frac{f(0)}{n-1}$$

for all $w \in T_{\gamma(0)}K$ *with* $g(w,w) = 1$, *where* $a > 0, c \geq 0$ *and* f *is a differentiable function with* $|f(t)| \leq c$.

Assume

$$L(\gamma) > \frac{\pi}{a}\left(\left(1 - \frac{k}{2(n-1)}\right)c + \sqrt{\left(1 - \frac{k}{2(n-1)}\right)^2 c^2 + a\left(n - 1 - \frac{3k}{4}\right)}\right).$$

Then γ *can not be maximal.*

Proof. Suppose that $\gamma : [0, L] \to M$ be a unit speed timelike geodesic with length L orthogonal at $\gamma(0)$ to the spacelike submanifold K. Set $E_n(t) = \gamma'(t)$ and let $\{E_1, E_2, \ldots\ldots, E_{n-1}\}$ be $n-1$ spacelike parallel fields such that $\{E_1(0), E_2(0), \ldots\ldots, E_k(0)\}$ forms an orthonormal basis of $T_{\gamma(0)}K$ and $\{E_1(t), E_2(t), \ldots\ldots, E_n(t)\}$ the orthonormal basis of $T_{\gamma(t)}M$. Set

$$W_i = \begin{cases} \cos(\frac{\pi t}{2L})E_i, & i = 1, 2, \ldots\ldots, k \\ \sin(\frac{\pi t}{L})E_i, & i = k+1, \ldots\ldots, n-1. \end{cases}$$

Then

$$W_i(0) = \begin{cases} E_i(0) \in T_{\gamma(0)}K, & i = 1, 2, \ldots\ldots, k \\ 0 \in T_{\gamma(0)}K, & i = k+1, \ldots\ldots, n-1. \end{cases}$$

Since $W_i(L) = 0$, $i = 1, 2, \ldots\ldots, n-1$, we have $W_i \in V_0^{\perp}(\gamma, K)$. Moreover, $|f(t)| \leq c$ implies $-c \leq \sin p(t) f(t) \leq c$ for any function p. Now, we compute the Lorentzian submanifold index form

$$\begin{aligned} I_{(b,K)}(W_i, W_i) =& g\left(S_{\gamma'(0)}W_i(0), W_i(0)\right) \\ &+ \int_0^L \left[g(R(W_i, \gamma')\gamma', W_i) - g(W_i', W_i')\right] dt \end{aligned}$$

as follows.

For i=1,2,.....,k,

$$
\begin{aligned}
&I_{(b,K)}(W_i, W_i)\\
&= g\left(S_{\gamma'(0)}E_i(0), E_i(0)\right)\\
&\quad + \int_0^L \left[\cos^2\left(\frac{\pi t}{2L}\right) g(R(E_i,\gamma')\gamma', E_i) - \left(\frac{\pi}{2L}\right)^2 \sin^2\left(\frac{\pi t}{2L}\right) g(E_i, E_i)\right] dt,\\
&\geq \frac{f(0)}{n-1} + \int_0^L \left[\cos^2\left(\frac{\pi t}{2L}\right)\frac{1}{n-1}\left(a + \frac{df}{dt}\right) - \left(\frac{\pi}{2L}\right)^2 \sin^2\left(\frac{\pi t}{2L}\right)\right] dt\\
&= \frac{f(0)}{n-1} + \frac{a}{n-1}\int_0^L \cos^2\left(\frac{\pi t}{2L}\right) dt + \frac{1}{n-1}\int_0^L \cos^2\left(\frac{\pi t}{2L}\right)\frac{df}{dt} dt\\
&\quad - \int_0^L \left(\frac{\pi}{2L}\right)^2 \sin^2\left(\frac{\pi t}{2L}\right) dt\\
&= \frac{f(0)}{n-1} + \frac{a}{n-1}\frac{L}{2}\\
&\quad + \frac{1}{n-1}\left[\cos^2\left(\frac{\pi t}{2L}\right) f(t)|_0^L + \int_0^L \left(\frac{\pi}{2L}\right)\sin\left(\frac{\pi t}{L}\right) f(t) dt\right] - \left(\frac{\pi}{2L}\right)^2 \frac{L}{2}\\
&\geq \frac{a}{n-1}\frac{L}{2} + \frac{1}{n-1}\frac{\pi}{2L}(-Lc) - \left(\frac{\pi}{2L}\right)^2 \frac{L}{2}.
\end{aligned}
$$

For i=k+1,......,n-1,

$$
\begin{aligned}
&I_{(b,K)}(W_i, W_i)\\
&= \int_0^L [\sin^2(\frac{\pi t}{L}) g(R(E_i,\gamma')\gamma', E_i) - (\frac{\pi}{L})^2 \cos^2(\frac{\pi t}{L}) g(E_i, E_i)] dt\\
&\geq \int_0^L [\sin^2(\frac{\pi t}{L})\frac{1}{n-1}(a + \frac{df}{dt}) - (\frac{\pi}{L})^2 \cos^2(\frac{\pi t}{L})] dt\\
&= \frac{a}{n-1}\int_0^L \sin^2(\frac{\pi t}{L}) dt + \frac{1}{n-1}\int_0^L \sin^2(\frac{\pi t}{L})\frac{df}{dt} dt\\
&\quad - \int_0^L (\frac{\pi}{L})^2 \cos^2(\frac{\pi t}{L}) dt\\
&= \frac{a}{n-1}\frac{L}{2} + \frac{1}{n-1}[\sin^2(\frac{\pi t}{L}) f(t)|_0^L - \int_0^L (\frac{\pi}{L})\sin(\frac{2\pi t}{L}) f(t) dt] - (\frac{\pi}{L})^2 \frac{L}{2}\\
&\geq \frac{a}{n-1}\frac{L}{2} + \frac{1}{n-1}(-(\frac{\pi}{L})Lc) - (\frac{\pi}{L})^2 \frac{L}{2}.
\end{aligned}
$$

Therefore, we have

$$\begin{aligned}
&\sum_{i=1}^{n-1} I_{(b,K)}(W_i, W_i)\\
\geq &\frac{aL}{2} - \pi c + \frac{k\pi c}{2(n-1)} - \frac{(n-1)\pi^2}{2L} + \frac{3k\pi^2}{8L}\\
= &\frac{2}{2L}\left[aL^2 - 2\left(1 - \frac{k}{2(n-1)}\right)\pi c L - \left(n - 1 - \frac{3k}{4}\right)\pi^2\right]\\
> &\, 0.
\end{aligned}$$

The last inequality is given by our hypothesis:

$$L > \frac{\pi}{a}\left(\left(1 - \frac{k}{2(n-1)}\right)c + \sqrt{\left(1 - \frac{k}{2(n-1)}\right)^2 c^2 + a\left(n - 1 - \frac{3k}{4}\right)}\right).$$

By Corollary 3.2, γ has a K-focal point. By Proposition 3.3, γ can not be maximal.

The following theorem is a Myers type diameter theorem. Set $diam_K(M, g) = \sup\{d(K, q) | q \in I^+(K)\}$.

THEOREM 4.2. *Let (M, g) be a globally hyperbolic space-time of dimension $n \geq 2$ and K the compact spacelike submanifold of dimension $k \geq 0$. Suppose there exist constants $a > 0$ and $c \geq 0$ such that for any point $q \in M$, and any unit maximal timelike geodesic γ in $\Omega_{K,q}$ with length L perpendicular at $\gamma(0)$ to K,*

$$g(R(u, \gamma'(t))\gamma'(t), u) \geq \frac{1}{n-1}\left(a + \frac{df}{dt}\right)$$

for all $u \in (\gamma'(t))^{\perp}$ with $g(u, u) = 1$ along γ

$$g(S_{\gamma'(0)}w, w) \geq \frac{f(0)}{n-1}$$

for all $w \in T_{\gamma(0)}K$ with $g(w, w) = 1$, where f is some function with $|f(t)| \leq c$ along γ. Then

$$diam_K(M,g) \leq \frac{\pi}{a}\left(\left(1-\frac{k}{2(n-1)}\right)c+\sqrt{\left(1-\frac{k}{2(n-1)}\right)^2 c^2+a\left(n-1-\frac{3k}{4}\right)}\right).$$

Proof. Suppose

$$diam_K(M,g) > \frac{\pi}{a}\left(\left(1-\frac{k}{2(n-1)}\right)c+\sqrt{\left(1-\frac{k}{2(n-1)}\right)^2 c^2+a\left(n-1-\frac{3k}{4}\right)}\right).$$

Then there exist a point q of M with $K << q$ such that

$$d(K,q) > \frac{\pi}{a}\left(\left(1-\frac{k}{2(n-1)}\right)c+\sqrt{\left(1-\frac{k}{2(n-1)}\right)^2 c^2+a\left(n-1-\frac{3k}{4}\right)}\right).$$

by definition of $diam_K(M,g)$.

Since M is a globally hyperbolic space-time, $d(K,q) > 0$ iff $q \in I^+(K)$, and since $J^-(q) \cap K$ is compact, by Proposition 3.4, thereis a timelike maximal geodesic γ perpendicular to K starting from K to q with

$$L = d(K,q) > \frac{\pi}{a}\left(\left(1-\frac{k}{2(n-1)}\right)c+\sqrt{\left(1-\frac{k}{2(n-1)}\right)^2 c^2+a\left(n-1-\frac{3k}{4}\right)}\right).$$

By Theorem 4.1, γ can not be maximal, in contradiction.

REMARK 4.3. (1) If $k = 0$, we obtain the same result of Theorem 2.6 and some-what a generalization of Theorem 4.1 in [ ].

(2) If K is any compact spacelike hypersurface of M, we have

$$diam_K(M, g) \leq \frac{\pi}{2a}\left(c + \sqrt{c^2 + (n-1)a}\right),$$

which is exactly a half of the upper bound of $diam(M, g)$ given in Theorem 2.6 and some-what a generalization of Theorem 4.1 in [ ].

(3) If K is a Cauchy hypersurface, i.e., a subset of M which every inextendible timelike curve intersects exactly once, Theorem 4.2 still holds.

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