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ON THE TENSOR PRODUCTS OF DUAL SPACES

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1. Introduction

For a Hilbert space $\mathcal{H}$ of an arbitrary dimension, let $\mathcal{L}(\mathcal{H})$ and $\mathcal{C}_1(\mathcal{H})$ denote the Banach spaces of all bounded linear operators and all trace class operators on $\mathcal{H}$, respectively. A *dual space* $\mathcal{M}$ in $\mathcal{L}(\mathcal{H})$ is, by definition, a σ-weakly closed subspace of $\mathcal{L}(\mathcal{H})$, and the *predual space* $\mathcal{M}_*$ of $\mathcal{M}$ is the Banach space of all σ-weakly continuous linear functionals on $\mathcal{M}$. A linear isomorphism between dual spaces is said to be a *dual space isomorphism* if it is bicontinuous with respect to the σ-weak topologies.

For two dual spaces $\mathcal{M} \subseteq \mathcal{L}(\mathcal{H})$ and $\mathcal{N} \subseteq \mathcal{L}(\mathcal{K})$, we denote by $\mathcal{M} \otimes \mathcal{N}$ the linear span of $\{A \otimes B : A \in \mathcal{M},\ B \in \mathcal{N}\}$ in $\mathcal{L}(\mathcal{H} \otimes \mathcal{K})$, where $\mathcal{H} \otimes \mathcal{K}$ denotes the Hilbert space tensor product of $\mathcal{H}$ and $\mathcal{K}$. The *tensor product* $\mathcal{M} \overline{\otimes} \mathcal{N}$ of $\mathcal{M}$ and $\mathcal{N}$ is then defined to be the σ-weak closure of $\mathcal{M} \otimes \mathcal{N}$ in $\mathcal{L}(\mathcal{H} \otimes \mathcal{K})$. In this note, we show that this tensor product of dual spaces is independent of the Hilbert spaces on which dual spaces act, in the following sense:

THEOREM 1.1. *Let $\mathcal{M}_i \subseteq \mathcal{L}(\mathcal{H}_i)$ and $\mathcal{N}_i \subseteq \mathcal{L}(\mathcal{K}_i)$ be dual spaces, and assume that there is a dual space isomorphism $\phi_i : \mathcal{M}_i \to \mathcal{N}_i$ for each $i = 1, 2$. Then there is a unique dual space isomorphism denoted by $\phi_1 \overline{\otimes} \phi_2 : \mathcal{M}_1 \overline{\otimes} \mathcal{M}_2 \to \mathcal{N}_1 \overline{\otimes} \mathcal{N}_2$ such that*

$$(1.1) \qquad (\phi_1 \overline{\otimes} \phi_2)(A \otimes B) = \phi_1(A) \otimes \phi_2(B), \qquad A \in \mathcal{M}_1,\ B \in \mathcal{M}_2.$$

First of all, we define a linear isomorphism from the algebraic tensor product $\mathcal{M}_* \otimes \mathcal{N}_*$ onto a dense subspace of $(\mathcal{M} \overline{\otimes} \mathcal{N})_*$, from which we get a norm-decreasing isomorphism from the projective Banach space

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tensor product $\mathcal{M}_* \otimes_\gamma \mathcal{N}_*$ onto $(\mathcal{M}\overline{\otimes}\mathcal{N})_*$. On the other hand, there exists a norm continuous isomorphism $\psi_i : \mathcal{N}_{i*} \to \mathcal{M}_{i*}$ such that $\psi_i^* = \phi_i$ for each $i = 1, 2$. We show that the map $\psi_1 \otimes_\gamma \psi_2$ induces the required dual space isomorphism.

2. Preliminaries

For a dual space $\mathcal{M}$ in $\mathcal{L}(\mathcal{H})$, the *preannihilator* ${}^\perp\mathcal{M}$ of $\mathcal{M}$ in $\mathcal{C}_1(\mathcal{H})$ is defined by

$$ {}^\perp\mathcal{M} = \{T \in \mathcal{C}_1(\mathcal{H}) : \mathrm{Tr}(TA) = 0 \quad \text{for all} \quad A \in \mathcal{M}\}, $$

where $\mathrm{Tr}(\cdot)$ denotes the canonical trace on $\mathcal{C}_1(\mathcal{H})$. We recall that there is an isometric linear isomorphism

$$ \mathcal{M}_* \ni \sigma \mapsto t \in \mathcal{C}_1(\mathcal{H})/{}^\perp\mathcal{M} \tag{2.1} $$

from $\mathcal{M}_*$ onto the quotient Banach space $\mathcal{C}_1(\mathcal{H})/{}^\perp\mathcal{M}$, which is uniquely determined under the following condition;

$$ \sigma(A) = \mathrm{Tr}(TA) \quad \text{for all} \quad A \in \mathcal{M}, \tag{2.2} $$

where T can be any fixed representative of the coset $t \in \mathcal{C}_1(\mathcal{H})/{}^\perp\mathcal{M}$ (see [**1**] for example).

For Banach spaces X and Y, we recall that the *projective cross norm* on the algebraic tensor product $X \otimes Y$ is defined by

$$ \|u\|_\gamma = \inf \sum_{i=1}^{n} \|x_i\| \|y_i\|, $$

where the infimum is taken over all expressions

$$ x = \sum_{i=1}^{n} x_i \otimes y_i, \qquad x_i \in X, \ y_i \in Y. $$

It turns out that $\|\cdot\|_\gamma$ is the largest one among cross norms. The completion of $X \otimes Y$ with respect to the norm $\|\cdot\|_\gamma$ is denoted by

$X \otimes_\gamma Y$. The projective cross norm $\|\cdot\|_\gamma$ on $X \otimes_\gamma Y$ is also described as follows:

$$\|u\|_\gamma = \inf \sum_{i=1}^{\infty} \|x_i\| \, \|y_i\|, \tag{2.3}$$

where the infimum is taken over all expressions $u = \sum_{i=1}^{\infty} x_i \otimes y_i$ for which convergence is considered in the projective cross norm $\|\cdot\|_\gamma$ on $X \otimes_\gamma Y$. (see [**2**, **6**] for example).

If $S \in \mathcal{C}_1(\mathcal{H})$ and $T \in \mathcal{C}_1(\mathcal{K})$ then it is easy to see that $S \otimes T \in \mathcal{C}_1(\mathcal{H} \otimes \mathcal{K})$ and

$$\mathrm{Tr}(S \otimes T) = \mathrm{Tr}(S)\mathrm{Tr}(T). \tag{2.4}$$

In this way, $\mathcal{C}_1(\mathcal{H}) \otimes \mathcal{C}_1(\mathcal{K})$ is a subspace of $\mathcal{C}_1(\mathcal{H} \otimes \mathcal{K})$. The following seems to be well-known and we omit the proof.

LEMMA 2.1. *We have the following:*

(i) $\mathcal{C}_1(\mathcal{H}) \otimes \mathcal{C}_1(\mathcal{K})$ *is a dense subspace of* $\mathcal{C}_1(\mathcal{H} \otimes \mathcal{K})$
(ii) $\mathcal{C}_1(\mathcal{H}) \otimes_\gamma \mathcal{C}_1(\mathcal{K})$ *is isometrically isomorphic to* $\mathcal{C}_1(\mathcal{H} \otimes \mathcal{K})$.

3. Preduals of the Tensor Products of Dual Spaces

Let $\mathcal{M} \subseteq \mathcal{L}(\mathcal{H})$ and $\mathcal{N} \subseteq \mathcal{L}(\mathcal{K})$ be dual spaces. For $\sigma \in \mathcal{M}_*$ and $\rho \in \mathcal{N}_*$, we define a linear functional $\sigma \otimes \rho$ on $\mathcal{M} \overline{\otimes} \mathcal{N}$ by

$$(\sigma \otimes \rho)(X) = \mathrm{Tr}\big((T_\sigma \otimes T_\rho)X\big), \qquad X \in \mathcal{M} \overline{\otimes} \mathcal{N},$$

where T_σ denotes a representative of the coset in $\mathcal{C}_1(\mathcal{H})/{}^\perp\mathcal{M}$ corresponding to σ satisfying (2.2), and similarly for ρ.

PROPOSITION 3.1. *For $\sigma \in \mathcal{M}_*$ and $\rho \in \mathcal{N}_*$, the functional $\sigma \otimes \rho$ defines an element of $(\mathcal{M} \overline{\otimes} \mathcal{N})_*$ and satisfies $\|\sigma\|\|\rho\| = \|\sigma \otimes \rho\|$.*

Proof. Let $T \in \mathcal{M}_*$ and $S \in \mathcal{N}_*$ such that

$$\mathrm{Tr}(TA) = \mathrm{Tr}(SB) = 0, \qquad A \in \mathcal{M},\ B \in \mathcal{N}.$$

Then we have

$$\mathrm{Tr}\big((T\otimes S)(A\otimes B)\big)=\mathrm{Tr}(TA\otimes SB)=\mathrm{Tr}(TA)\mathrm{Tr}(SB)=0$$

for each $A\in\mathcal{M}$ and $B\in\mathcal{N}$ by (2.4). This shows that

$$\mathrm{Tr}\big((T\otimes S)X\big)=0,\qquad X\in\mathcal{M}\overline{\otimes}\mathcal{N}.$$

Hence, $\sigma\otimes\rho$ defines a well-defined σ-weak continuous functional on $\mathcal{M}\overline{\otimes}\mathcal{N}$.

Note that $\|\sigma\otimes\rho\|=\|[T_\sigma\otimes T_\rho]\|$, where $[T_\sigma\otimes T_\rho]$ denotes the coset in $\mathcal{C}_1(\mathcal{H}\otimes\mathcal{K})/{}^\perp(\mathcal{M}\overline{\otimes}\mathcal{N})$. For any $X\in{}^\perp\mathcal{M}$ and $Y\in{}^\perp\mathcal{N}$, we have

$$\|T_\sigma+X\|_1\|T_\rho+Y\|_1=\|(T_\sigma+X)\otimes(T_\rho+Y)\|_1=\|(T_\sigma\otimes T_\rho)+R\|_1,$$

where $R=(X\otimes T_\rho)+(T_\sigma\otimes Y)+(X\otimes Y)\in{}^\perp(\mathcal{M}\overline{\otimes}\mathcal{N})$ and $\|\cdot\|_1$ denotes the trace class norm. Hence, it follows that $\|[T_\sigma\otimes T_\rho]\|\le\|[T_\sigma]\|\,\|[T_\rho]\|$, and we see

$$\|\sigma\otimes\rho\|\le\|\sigma\|\,\|\rho\|.$$

For the reverse inequality, we take A_0 and B_0 in the unit ball of $\mathcal{M}$ and $\mathcal{N}$, respectively, such that $\|\sigma\|=|\sigma(A_0)|$ and $\|\rho\|=|\rho(B_0)|$. Then, by (2.4), we have

$$\begin{aligned}\|\sigma\|\,\|\rho\|&=|\sigma(A_0)\sigma(B_0)|=|\mathrm{Tr}(T_\sigma A_0)\mathrm{Tr}(T_\rho B_0)|\\&=|\mathrm{Tr}(T_\sigma\otimes T_\rho)(A_0\otimes B_0)|=|(\sigma\otimes\rho)(A_0\otimes B_0)|\le\|\sigma\otimes\rho\|,\end{aligned}$$

because $\|A_0\otimes B_0\|=\|A_0\|\|B_0\|\le 1$. □

From Proposition 3.1, we have a unique linear map $\alpha:\mathcal{M}_*\otimes\mathcal{N}_*\to(\mathcal{M}\overline{\otimes}\mathcal{N})_*$ such that the following diagram commutes:

$$(3.1)\qquad \begin{array}{ccc}\mathcal{C}_1(\mathcal{H})\otimes\mathcal{C}_1(\mathcal{K}) & \longrightarrow & \mathcal{C}_1(\mathcal{H}\otimes\mathcal{K})\\ \big\downarrow & & \big\downarrow\\ \mathcal{M}_*\otimes\mathcal{N}_* & \xrightarrow{\ \alpha\ } & (\mathcal{M}\overline{\otimes}\mathcal{N})_*\end{array}$$

where the vertical arrows denote the quotient maps in (2.1).

PROPOSITION 3.2. *The above linear map α is an isomorphism whose range is dense in $(\mathcal{M}\overline{\otimes}\mathcal{N})_*$.*

Proof. Let $\phi = \sum_{i=1}^n \sigma_i \otimes \rho_i$ be a zero element of $(\mathcal{M}\overline{\otimes}\mathcal{N})_*$, where $\{\rho_i\}$ is linearly independent in $\mathcal{N}_*$. We choose $\{B_j : j = 1, 2, \dots, n\}$ in $\mathcal{N}$ such that $\mathrm{Tr}(T_{\rho_i} B_j) = \delta_{ij}$, the Kronecker delta. Then, for any $A \in \mathcal{M}$, we have

$$0 = \sum_{i=1}^n \mathrm{Tr}\big((T_{\sigma_i} \otimes T_{\rho_i})(A \otimes B_j)\big) = \sum_{i=1}^n \mathrm{Tr}(T_{\sigma_i} A)\mathrm{Tr}(T_{\rho_i} B_j) = \mathrm{Tr}(T_{\sigma_j} A),$$

for each $j = 1, 2, \dots, n$. Hence, we have $\sigma_j = 0$ for each $j = 1, 2, \dots, n$, and so, $\phi = 0$ in the algebraic tensor product $\mathcal{M}_* \otimes \mathcal{N}_*$. The density of the image follows from the diagram (3.1) and Lemma 2.1.(i). □

By Proposition 3.2, the map α in (3.1) extends to a norm-decreasing linear map

$$\alpha : \mathcal{M}_* \otimes_\gamma \mathcal{N}_* \to (\mathcal{M}\overline{\otimes}\mathcal{N})_* \tag{3.2}$$

on $\mathcal{M}_* \otimes_\gamma \mathcal{N}_*$ which is still denoted by α. From (2.3), we see that every element in $(\mathcal{M}\overline{\otimes}\mathcal{N})_*$ may be expressed in the form $\sum_{i=1}^\infty \sigma_i \otimes \rho_i$ with $\sigma_i \in \mathcal{M}_*$ and $\rho \in \mathcal{N}_*$ in the norm convergence, and so we have

$$\Big(\sum_{i=1}^\infty \sigma_i \otimes \rho_i\Big)(A \otimes B) = \sum_{i=1}^\infty \sigma_i(A)\rho_i(B), \qquad A \in \mathcal{M},\ B \in \mathcal{N}. \tag{3.3}$$

PROPOSITION 3.3. *Let $\mathcal{M}_i$ and $\mathcal{N}_i$ be dual spaces. If $\psi_i : \mathcal{M}_{i*} \to \mathcal{N}_{i*}$ be a continuous linear map with respect to the norm topologies for $i = 1, 2$, then there is a unique norm-continuous linear map*

$$\psi_1 \otimes_* \psi_2 : (\mathcal{M}_1\overline{\otimes}\mathcal{M}_2)_* \to (\mathcal{N}_1\overline{\otimes}\mathcal{N}_2)_*$$

such that the following diagram commutes:

$$\begin{array}{ccc}
\mathcal{M}_{1*} \otimes_\gamma \mathcal{M}_{2*} & \xrightarrow{\psi_1 \otimes_\gamma \psi_2} & \mathcal{N}_{1*} \otimes_\gamma \mathcal{N}_{2*} \\
{\scriptstyle \alpha_1}\downarrow & & \downarrow{\scriptstyle \alpha_2} \\
(\mathcal{M}_1\overline{\otimes}\mathcal{M}_2)_* & \xrightarrow{\psi_1 \otimes_* \psi_2} & (\mathcal{N}_1\overline{\otimes}\mathcal{N}_2)_*
\end{array}$$

where α_i is the map in (3.2) for $i = 1, 2$.

Proof. It suffices to show that

$$(\psi_1 \otimes_\gamma \psi_2)(\ker \alpha_1) = \ker \alpha_2.$$

Let $u = \sum_i \sigma_i \otimes \rho_i \in \mathcal{M}_{1*} \otimes_\gamma \mathcal{M}_{2*}$ and $\alpha_1(u) = 0$. We denote the adjoint map by $\psi_i^* : \mathcal{N}_i \to \mathcal{M}_i$ for $i = 1, 2$. Now, for every $A \in \mathcal{N}_1$ and $B \in \mathcal{N}_2$, we have by (3.3) that

$$\begin{aligned}
\big((\psi_1 \otimes_\gamma \psi_2)(u)\big)(A \otimes B) &= \Big(\sum_{i=1}^{\infty} \psi_1(\sigma_i) \otimes \psi_2(\rho_i)\Big)(A \otimes B) \\
&= \sum_{i=1}^{\infty} \psi_1(\sigma_i)(A)\psi_1(\rho_i)(B) \\
&= \sum_{i=1}^{\infty} \sigma_i(\psi_1^* A)\rho_i(\psi_2^* B) \\
&= \Big(\sum_{i=1}^{\infty} \sigma_i \otimes \rho_i\Big)(\psi_1^* A \otimes \psi_2^* B) = 0,
\end{aligned}$$

because $\psi_1^* A \in \mathcal{M}_1$, $\psi_2^* B \in \mathcal{M}_2$ and $\sum_i \sigma_i \otimes \rho_i = 0$ on $(\mathcal{M}_1 \overline{\otimes} \mathcal{M}_2)_*$. This shows that $(\psi_1 \otimes_\gamma \psi_2)(\ker \alpha_1) \subseteq \ker \alpha_2$. The reverse inclusion is similar. □

Proof of Theorem 1.1. The uniqueness is clear. Because $\phi_i : \mathcal{M}_i \to \mathcal{N}_i$ is a dual space isomorphism, there is a norm continuous isomorphism $\psi : \mathcal{N}_{i*} \to \mathcal{M}_{i*}$ such that $\psi_i^* = \phi_i$ for each $i = 1, 2$. By Proposition 3.3, we have a norm continuous isomorphism $\psi_1 \otimes_* \psi_2 : (\mathcal{N}_1 \overline{\otimes} \mathcal{N}_2)_* \to (\mathcal{M}_1 \overline{\otimes} \mathcal{M}_2)_*$. It is easy to check that $(\psi_1 \otimes_* \psi_2)^*$ is the required dual space isomorphism. □

This note is a full version of [4]. The authors are grateful to Professor I. B. Jung who brought us the defining problem of the tensor products of dual algebras [1] and related questions. After completing this note, the authors received a preprint by Ruan [5], in which he obtained a similar result on the preduals of the tensor products of *dual operator spaces*. Especially, he showed that the map α in (3.2) is actually a complete contraction [5, Theorem 3.4]. We also note that the map α

need not to be an isometry by [**5**, Proposition 3.3] and [**3**, Example 3.3], although it is an isometry if $\mathcal{M} = \mathcal{L}(\mathcal{H})$ and $\mathcal{N} = \mathcal{L}(\mathcal{K})$ by Lemma 2.1(ii). Actually, α is an isometry if $\mathcal{M}$ and $\mathcal{N}$ are von Neumann algebras [**5**, Corollary 3.6].

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