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SOME COMPUTATIONS OF RELATIVE NIELSEN NUMBERS

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1. Introduction

H. Schirmer introduced the relative Nielsen number $N(f;X,A)$ in [8] which is a lower bound for the number of fixed points for all maps in the relative homotopy class of f. In [10] the Nielsen number of the boundary $\tilde{n}(f;X,A)$ is a lower bound for the number of fixed points on the boundary of A denoted Bd A only when any selfmap of (X,A) has a minimal fixed point set and the Nielsen number of the complement $\widetilde{N}(f;X,A)$, that is, the number of fixed point classes of $f:X\to X$ which do not assume their index in A, is a lower bound for the number of fixed points on $Cl(X-A)$.

In the classical setting, where $A=\phi$, $f_\pi(\pi_1(X))\subset J(f)$ the trace subgroup of cyclic homotopies and $R(f)$ the Reidemeister number of f introduced in section 3, it follows from $L(f)\neq 0$ that $N(f)=R(f)$ [5, p.33, Theorem 4.2]. It is a purpose of this paper to generalize this fact to maps of pairs of spaces (Theorem 3.1 and 3.2). In section 2, $n(f;X-A)$ will be defined and we will show that maps in the homotopy class of f which have a $N(f;X-A)$ fixed points on $X-A$ must have at least $n(f;X-A)$ fixed points on Bd A, and we will calculate $n(f;X-A)$ in some special cases. In section 3, methods to compute relative Nielsen numbers with relative Lefschetz numbers are given. Throughout this paper, $f:(X,A)\to(X,A)$ will be a selfmap of a pair of compact polyhedra with X connected and we will follow the notations and terminology of [10].

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2. Relative Nielsen numbers

Let $f : (X, A) \to (X, A)$ be a selfmap of a pair of compact polyhedra. We shall write $\bar{f} : A \to A$ for the restriction of f to A and write $f : X \to X$ if the condition that $f(A) \subset A$ is immaterial. Let $\hat{A} = \bigcup_1^n A_k$ be the disjoint union of all components of A which are mapped by f into themselves, and we shall write $f_k : A_k \to A_k$ for the restriction of f to A_k. We write Fix f for the fixed point set $\{x \in X | f(x) = x\}$ and F for a fixed point class of $f : X \to X$. A fixed point class F of $f : X \to X$ is a *weakly common fixed point class of* f *and* $\bar{f}$ if it contains a fixed point class of $f_k : A_k \to A_k$ for some k [13]. H. Schirmer defined the fixed point class F of $f : X \to X$ *assumes its index in* A if

$$ind(X, f, F) = ind(A, \bar{f}, F \cap A).$$

Also she defined the relative Nielsen number $\tilde{n}(f; X, A)$ by the number of common fixed point classes of $f : X \to X$ which do not assume their index in A [10].

DEFINITION 2.1. The number of weakly common fixed point classes of $f : X \to X$ which do not assume their index in A is called the relative Nielsen number of the boundary space Bd A, denoted $n(f; X - A)$.

It is clear $n(f; X - A) \geq \tilde{n}(f; X, A)$ by the definition. In general, $n(f; X - A)$ is different from $\tilde{n}(f; X, A)$. As a simple example, if we take the identity map $f : (B^2, S^1) \to (B^2, S^1)$ of the pair of a 2-dimensional ball and its boundary, then $\tilde{n}(f; X, A) = 0$, but $n(f; X - A) = 1$. If all the fixed point classes of $\bar{f}$ are essential, then $n(f; X - A) = \tilde{n}(f; X, A)$.

In [13], $N(f; X - A)$ is defined by the number of essential fixed point classes of $f : X \to X$ which are not weakly common fixed point classes, and $E(f, \bar{f})$ is defined by the number of essential weakly common fixed point classes of f and $\bar{f}$

THEOREM 2.1. *If* $f : (X, A) \to (X, A)$ *is a map, then* $\tilde{N}(f; X, A) = n(f; X - A) + N(f) - E(f, \bar{f})$ *and hence* $n(f; X - A) = \tilde{N}(f; X, A) - N(f; X - A)$.

Proof. Let F be a fixed point class of $f : X \to X$. A fixed point class F which is not a weakly common fixed point class of f and $\bar{f}$ is essential if and only if it does not assume its index in A, and so we get $n(f;X-A)+[N(f)-E(f,\bar{f})] = \sharp\ \{\ F|F$ is a weakly common fixed point class of f and $\bar{f}$ and does not assume its index in $A\ \} + \sharp\ \{\ F|F$ is not a weakly common fixed point class of f and $\bar{f}$ which is essential $\} = \sharp\ \{\ F|F$ is a weakly common fixed point class of f and $\bar{f}$ and does not assume its index in $A\ \} + \sharp\ \{F|F$ is not a weakly common fixed point class of f and $\bar{f}$ and does not assume its index in $A\ \} = \widetilde{N}(f;X,A)$.

$N(f;X-A)+E(f,\bar{f}) = N(f)$ shows the second part of the theorem.

The lower bound property of $n(f;X-A)$ follows immediately from Theorem 2.1.

THEOREM 2.2(LOWER BOUND). *Any map $f:(X,A)\to(X,A)$ which has $N(f;X-A)$ fixed points on $X-A$ has at least $n(f;X-A)$ fixed points on Bd A.*

EXAMPLE 2.1. Let $X=B^{n+1}$ be the unit ball $\{x\in R^{n+1}|\|x\|\le 1\}$ in R^{n+1} for $n\ge 2$, $A=\{x\in X|1/2\le\|x\|\le 1\}$, and $f:(X,A)\to(X,A)$ be the identity, then

$$N(\bar{f}) = \begin{cases} 0, & \text{if } n \text{ is odd} \\ 1, & \text{if } n \text{ is even,} \end{cases}$$

$$\tilde{N}(f;X,A) = n(f;X-A) = 1.$$

By [9, Theorem 4.1], there exists a deformation $g:(X,A)\to(X,A)$ such that if n is odd, then g has $N(f;X,A)$ fixed point on $X-A$ and no further fixed point.

3. Main results

Pick a base point $a_k\in A_k$ for each $A_k\subset\hat{A}$ and a base point $x_0\in X$. It is well known that the covering translations of universal covering spaces $\tilde{A}_k$ and $\tilde{X}$ of A_k and X form groups $\mathcal{D}_k=\mathcal{D}_k(\tilde{A}_k,p_k)$ and $\mathcal{D}=\mathcal{D}(\tilde{X},p)$ which are isomorphic to $\pi_1(A_k)$ and $\pi_1(X)$ respectively. Recall that points of $\tilde{A}_k$ and $\tilde{X}$ are respectively in one-to-one correspondence with the path classes in A_k and X starting from a_k and x_0. Under this identification, let $\tilde{a}_k=\langle e_k\rangle\in\tilde{A}_k$ and $\tilde{x}_0=\langle e\rangle\in\tilde{X}$ be the constant paths. Pick a path w_k in A_k from a_k to $f_k(a_k)$ for each k and a path

w_0 in X from x_0 to $f(x_0)$. Then there are unique liftings $\tilde{f}_k$ and $\tilde{f}$ of maps $f_k : A_k \to A_k$ and $f : X \to X$ such that $\tilde{f}_k(\langle e_k \rangle) = \langle w_k \rangle \in \tilde{A}_k$ and $\tilde{f}(\langle e \rangle) = \langle w_0 \rangle \in \tilde{X}$. Let liftings $\tilde{f}_k$ and $\tilde{f}$ be chosen as references, then the endomorphism $\tilde{f}_\pi : \mathcal{D} \to \mathcal{D}$ determined by a lifting $\tilde{f}$ of f is defined by

$$\tilde{f}_\pi(\alpha) \circ \tilde{f} = \tilde{f} \circ \alpha, \alpha \in \mathcal{D},$$

and the $\tilde{f}_\pi$-conjugacy class of $\gamma \in \pi$ is said to be the coordinate of a fixed point class $pFix(\gamma \circ \tilde{f})$ [5]. The coordinate of a fixed point class can be obtained geometrically.

LEMMA 3.1. *The coordinate for the class of a fixed point x of f is the $\tilde{f}_\pi$-conjugacy class of $\gamma = \langle c(f \circ c)^{-1} w_0^{-1} \rangle \in \pi$, where c is any path from x_0 to x. In other words, $x \in pFix(\gamma \circ \tilde{f})$.*

Proof. Let $\tilde{x} = \langle c \rangle \in p^{-1}(x)$. Since $\tilde{f}(\tilde{x}_0) = \tilde{f}(\langle e \rangle) = \langle w_0 \rangle$, we have $\tilde{f}(\tilde{x}) = \tilde{f}(\langle c \rangle) = \langle w_0(f \circ c) \rangle$. Hence $(\gamma \circ \tilde{f})(\tilde{x}) = \gamma \langle w_0(f \circ c) \rangle = \langle c(f \circ c)^{-1} w_0^{-1} \rangle \langle w_0(f \circ c) \rangle = \tilde{x}$.

Let $f : X \to X$ be a givn selfmap. The set of fixed points of f is denoted by $\Phi(f)$ instead of Fix f. Two fixed points $x, y \in \Phi(f)$ are said to be equivalent if x and y belong to the same fixed point class, i.e., if there exists a path $\lambda : I \to X$ such that $\lambda(0) = x, \lambda(1) = y$ and λ is homotopic to $f \circ \lambda$ rel. end points . We denote by $\Phi(f)/\sim$ the set of equivalence classes of $\Phi(f)$ by this equivalence relation. Let $F \in \Phi(f)/\sim$ and $x \in F$ be given. Define $\tau(F)$ as the unique class of FPC(f) determined by x where FPC(f) is the fixed point class data of f, the weighted set of lifting classes of f, the weight of a class $[\tilde{f}]$ being $ind(X, f, pFix\tilde{f})$ [5, Ch. III, Sec. 1]. This correspondence gives a well-defined function $\tau : \Phi(f)/\sim \longrightarrow$FPC$(f)$. Also we can define $\mu_k : \Phi(f_k)/\sim \longrightarrow \Phi(f)/\sim$ by $\mu_k(F_k) = F$ determined by $x_k \in F_k \subset F$, and thus we have a commutative diagram

$$\begin{array}{ccc} \Phi(f_k)/\sim & \xrightarrow{\tau_k} & \mathrm{FPC}(f_k) \\ {\scriptstyle \mu_k}\downarrow & & \downarrow{\scriptstyle i_{k,\mathrm{FPC}}} \\ \Phi(f)/\sim & \xrightarrow[\tau]{} & \mathrm{FPC}(f). \end{array}$$

Note that we shall fail to distinguish between a path in X and its class in the fundamental groupoid of X. In [4], the group homomorphism $f^{w_0} : \pi_1(X, x_0) \to \pi_1(X, x_0)$ defined by $f^{w_0}(\alpha) = w_0 f(\alpha) w_0^{-1}$ for every $\alpha \in \pi$ defines an equivalence relation on π by setting $\alpha \sim \alpha'$ if there exists a $\beta \in \pi$ such that $\alpha = \beta\alpha' f^{w_0}(\beta^{-1})$. Let $Coker\ (1 - f^{w_0})$ be the quotient set of π by this equivalence relation. The Reidemeister number of f is the number $R(f) = \sharp\ Coker\ (1 - f^{w_0})$. In what follows, $j : \pi \to Coker\ (1 - f^{w_0})$ denotes the quotient funtion: if $\langle\alpha\rangle \in \pi$, then $j(\langle\alpha\rangle) = [\langle\alpha\rangle] \in Coker\ (1 - f^{w_0})$.

Pick a path u_k from x_0 to a_k and take a lifting $\tilde{i}_k$ of i_k such that $\tilde{i}_k(\langle e_k\rangle) = \langle u_k\rangle$. Define a function $\nu_{k,\pi} : \pi_1(A_k, a_k) \to \pi_1(X, x_0)$ by

$$\nu_{k,\pi}\langle\alpha\rangle = \langle u_k(i_k \circ \alpha) w_k (f \circ u_k)^{-1} w_0^{-1}\rangle.$$

LEMMA 3.2. *The function $\nu_{k,\pi}$ induces a transformation*

$$\nu_k : Coker\ (1 - f_k^{w_k}) \to Coker\ (1 - f^{w_0})$$

and ν_k is independent of the choice of the path u_k.

Proof. See [12, Lemma 1.2].

LEMMA 3.3. *The diagram*

$$\begin{array}{ccc} \Phi(f_k)/\sim & \xrightarrow{\rho_k} & Coker\ (1 - f_k^{w_k}) \\ \downarrow \mu_k & & \downarrow \nu_k \\ \Phi(f)/\sim & \xrightarrow[\rho]{} & Coker\ (1 - f^{w_0}) \end{array}$$

commutes, where $\rho(F) = \big[\langle c(f \circ c)^{-1} w_0^{-1}\rangle\big]$, c is any path in X with $c(0) = x_0, c(1) = x$, for any $x \in F$.

Proof. Let $x_k \in F_k \in \Phi(f_k)/\sim$ and pick a path c_k from a_k to x_k in A_k. Since ρ is independent of the choice of the path c, pick a path c from x_0 to $x_k \in F$ (as $F_k \subset F$) in X. By Lemma 3.2, $\nu_k\big[\langle\alpha\rangle\big] = \big[\langle u_k(i_k \circ \alpha) w_k (f \circ u_k)^{-1} w_0^{-1}\rangle\big]$, we have

$$\begin{aligned} \nu_k \rho_k(F_k) &= \nu_k\big[\langle c_k(f_k \circ c_k)^{-1} w_k^{-1}\rangle\big] \\ &= \big[\langle u_k c_k (f_k \circ c_k)^{-1} w_k^{-1} w_k (f \circ u_k)^{-1} w_0^{-1}\rangle\big] \\ &= \big[\langle u_k c_k (f \circ (u_k c_k))^{-1} w_0^{-1}\rangle\big] \\ &= \rho(F) \\ &= \rho\mu_k(F_k). \end{aligned}$$

We recall two lemmas (see [4, Lemma A.1, A.2]).

LEMMA 3.4. *Let $f : X \to X, x_0 \in X$ and w_0 and η be paths in X connecting x_0 to $f(x_0)$. Then, there is an index preserving bijection $r_{w_0,\eta}$: Coker $(1 - f^{w_0}) \to$ Coker $(1 - f^{\eta})$ given by $r_{w_0,\eta}[\langle\alpha\rangle] = [\langle\alpha w_0 \eta^{-1}\rangle]$.*

LEMMA 3.5. *Let $f : X \to X, x_0 \in X$ and $w_0 : I \to X$ be given, with $w_0(0) = x_0, w_0(1) = f(x_0)$. Let $a_k \in A_k \subset X$ be another base point and let $u_k : I \to X$ be a path in X connecting x_0 to a_k. Then, $u_{k*} = u_*$: Coker $(1 - f^{w_0}) \to$ Coker $(1 - f^{u_k^{-1}w_0(f\circ u_k)})$ defined by $u_*[\langle\alpha\rangle] = [\langle u_k^{-1}\alpha u_k\rangle]$ is an index preserving bijection.*

Consider the commutative diagram

$$\begin{array}{ccc} \pi_1(A_k, a_k) & \xrightarrow{f_k^{w_k}} & \pi_1(A_k, a_k) \\ {\scriptstyle i_{k,\pi}}\big\downarrow & & \big\downarrow{\scriptstyle i_{k,\pi}} \\ \pi_1(X, a_k) & \xrightarrow[f^{w_k}]{} & \pi_1(X, a_k). \end{array}$$

If $i_{k,\pi}$ is surjective, then we have an exact sequence

$$0 \longrightarrow Ker\ i_{k,\pi} \longrightarrow \pi_1(A_k, a_k) \xrightarrow{i_{k,\pi}} \pi_1(X, a_k) \longrightarrow 0.$$

LEMMA 3.6. *If $i_{k,\pi}$ is surjective and the restriction $f_k^{w_k}|_{Ker\ i_{k,\pi}}$ of $f_k^{w_k}$ to $Ker\ i_{k,\pi}$ is nilpotent, then*

$$\nu_k : Coker\ (1 - f_k^{w_k}) \longrightarrow Coker\ (1 - f^{w_0})$$

is bijective.

Proof. Applying [3, Proposition 1.11], $i_{k,\pi}$ induces a bijection

$$i_k : Coker\ (1 - f_k^{w_k}) \to Coker\ (1 - f^{w_k})$$

defined by $i_k[\langle\alpha\rangle] = [\langle i_k \circ \alpha\rangle]$.

With u_k as above, define $\eta = u_k^{-1} w_0 (f \circ u_k)$. Then, by Lemma 3.4 and 3.5, it suffices to check that the diagram

$$\begin{array}{ccc} Coker\ (1-f_k^{w_k}) & \xrightarrow{i_k} & Coker\ (1-f^{w_k}) \\ \downarrow \nu_k & & \downarrow r_{w_k,\eta} \\ Coker\ (1-f^{w_0}) & \xrightarrow{u_*} & Coker\ (1-f^{u_k^{-1}w_0(f\circ u_k)}) \end{array}$$

commutes.

Let $[\langle\alpha\rangle] \in Coker\ (1-f_k^{w_k})$, then

$$\begin{aligned} r_{w_k,\eta} i_k[\langle\alpha\rangle] &= r_{w_k,\eta}[\langle i_k \circ \alpha\rangle] \\ &= [\langle (i_k\circ\alpha)w_k\eta^{-1}\rangle] \\ &= [\langle (i_k\circ\alpha)w_k(f\circ u_k)^{-1}w_0^{-1}u_k\rangle] \end{aligned}$$

and

$$\begin{aligned} u_k\nu_k[\langle\alpha\rangle] &= u_k[\langle u_k(i_k\circ\alpha)w_k(f\circ u_k)^{-1}w_0^{-1}\rangle] \\ &= [\langle u_k^{-1}u_k(i_k\circ\alpha)w_k(f\circ u_k)^{-1}w_0^{-1}u_k\rangle] \\ &= [\langle (i_k\circ\alpha)w_k(f\circ u_k)^{-1}w_0^{-1}u_k\rangle]. \end{aligned}$$

Then we have $r_{w_k,\eta} i_k[\langle\alpha\rangle] = u_k\nu_k[\langle\alpha\rangle]$.

Recall the relative Lefschetz number $L(f|_{(X,A)}) = L(f) - L(\bar{f})$ of $f:(X,A)\to(X,A)$ and the trace subgroup of cyclic homotopies

$$\begin{aligned} J(f,x_0) = \{\xi \in \pi_1(X,f(x_0)) \mid &\text{ there exists a homotopy} \\ &H: f\simeq f: X\times I \to X \ni \langle H(x_0,\ )\rangle = \xi\}. \end{aligned}$$

In [5, p.33, Theorem 4.2] where $f_\pi(\pi_1(X)) \subset J(f)$, it follows from $L(f)\neq 0$ that $N(f) = R(f)$. We prove the main theorems.

THEOREM 3.1. *Let $f:(X,A)\to(X,A)$ be a selfmap of a pair of compact polyhedra with $\hat{A} = \cup_{k=1}^n A_k$. If $f_\pi(\pi_1(X)) \subset J(f)$, $f_{k,\pi}(\pi_1(A_k)) \subset J(f_k)$, $i_{k,\pi}$ is surjective and $f_k^{w_k}|_{Ker\ i_{k,\pi}}$ is nilpotent for all k, then*

$$n(f;X-A) = \begin{cases} \sharp\ Coker\ (1-f^{w_0}), & \text{if } L(f|_{(X,A)}) \neq 0 \\ 0, & \text{otherwise.} \end{cases}$$

Proof. If $L(f_k) = 0$ for all k, then this theorem is clear. We can assume that $L(f_K) \neq 0$ for all $k, 1 \leq k \leq m$ for some $m \leq n$. By [5, p.33, Theorem 4.2], when $f_{k,\pi}(\pi_1(A_k)) \subset J(f_k)$ for all k, $1 \leq k \leq m$, the correspondence ρ_k is bijective. Let F be a fixed point class of $f : X \to X$. Then

$$\begin{aligned} ind(A, \bar{f}, F \cap A) &= ind(A, \bar{f}, \cup_{k=1}^{n}(F \cap A_k)) \\ &= \sum_{k=1}^{n} ind(A_k, f_k, F \cap A_k) \\ &= \sum_{k=1}^{m} ind(A_k, f_k, F_k) \quad \text{(by Lemma 3.3, 3.6)} \end{aligned}$$

for some fixed $F_k \in \Phi(f_k)/\sim$.

Case 1) Suppose $L(f) = 0$. Then all the fixed point classes of $f : X \to X$ are inessential. If $L(f|_{(X,A)}) \neq 0$, then there exists a component A_k such that

$$ind(A, \bar{f}, F \cap A) = \sum_{k=1}^{m} ind(A_k, f_k, F_k) = L(\bar{f})/N(f_k) \neq 0.$$

Hence all the fixed point classes of f do not assume their index in A. If $L(f|_{(X,A)}) = 0$, then $L(\bar{f}) = 0$, and so

$$ind(A, \bar{f}, F \cap A) = \sum_{k=1}^{m} ind(A_k, f_k, F_k) = 0 = ind(X, f, F)$$

because F is inessential. Thus all the fixed point classes of f assume their index in A.

Case 2) Suppose $L(f) \neq 0$. By using [5, p.33, Theorem 4.2] again, $N(f) > 0$. Thus we have

$$ind(A, \bar{f}, F \cap A) = L(\bar{f})/N(f)$$

and

$$ind(X, f, F) = L(f)/N(f).$$

This completes the theorem.

If $n = 1$, i.e. $\hat{A}$ is connected, we can take $w_0 = w_1$ and $x_0 = a_1$. Then $\nu_1\left[\langle\alpha\rangle\right] = \left[\langle i_1 \circ \alpha\rangle\right]$ and $\nu_1 = i_* : Coker\ (1 - f_1^{w_0}) \to Coker\ (1 - f^{w_0})$. We shall get

COROLLARY 3.1. *Let $f : (X, A) \to (X, A)$ be a selfmap of a pair of compact polyhedra with $\hat{A}$ connected. If $f_\pi(\pi_1(X)) \subset J(f), f_{1,\pi}(\pi_1(A_1)) \subset J(f_1)$, $i_{1,\pi}$ is surjective and $f_1^{w_0}|_{Ker\ i_{1,\pi}}$ is nilpotent, then*

$$n(f; X - A) = \begin{cases} \sharp\ Coker\ (1 - f^{w_0}), & \text{if } L(f_1) \neq L(f) \\ 0, & \text{otherwise.} \end{cases}$$

In [13], X.Zhao showed that if there is a component A_k of $\hat{A}$ such that $i_{k,\pi}$ is surjective, then $N(f; X - A) = 0$. By Theorem 2.1 and Theorem 3.1, we have

THEOREM 3.2. *Let $f : (X, A) \to (X, A)$ be a selfmap of a pair of compact polyhedra with $\hat{A} = \cup_{k=1}^n A_k$. Suppose $f_\pi(\pi_1(X)) \subset J(f)$, $f_{k,\pi}(\pi_1(A_k)) \subset J(f_k)$, $i_{k,\pi}$ is surjective and $f_k^{w_k}|_{Ker\ i_{k,\pi}}$ is nilpotent for all k, then*

$$\tilde{N}(f; X, A) = \begin{cases} \sharp\ Coker\ (1 - f^{w_0}), & \text{if } L(f|_{(X,A)}) \neq 0 \\ 0, & \text{otherwise.} \end{cases}$$

COROLLARY 3.2. *Let $f : (X, A) \to (X, A)$ be a selfmap of a pair of compact polyhedra with $\hat{A}$ connected. If $f_\pi(\pi_1(X)) \subset J(f), f_{1,\pi}(\pi_1(A_1)) \subset J(f_1)$, $i_{1,\pi}$ is surjective and $f_1^{w_0}|_{Ker\ i_{1,\pi}}$ is nilpotent, then*

$$\tilde{N}(f; X, A) = \begin{cases} \sharp\ Coker\ (1 - f^{w_0}), & \text{if } L(f_1) \neq L(f) \\ 0, & \text{otherwise.} \end{cases}$$

EXAMPLE 3.1. Let $X = \{x \in R^2 | 1/2 \leq \|x\| \leq 1\}$ be an annulus in R^2 and let $A = \cup_{k=1}^2 A_k$ be the boundary of X where $A_k = \{x \in X | \|x\| = 1/k\}$. Define $f : (X, A) \to (X, A)$ by $f(re^{i\theta}) = re^{i3\theta}$ for $1/2 \leq r \leq 1$. Take $e^{i0} = 1$ as base point of X and choose the path w_0 to be constant. Then, for all n, $1 - f^n : \mathbf{Z} \to \mathbf{Z}$ is multiplication by $1 - 3^n$ and

$$Coker\ (1 - f^n) = \mathbf{Z}_{3^n - 1}.$$

Since A_k's are H-spaces, $L(f_k^n) = L(f^n)$ for each k. Then we have

$$\tilde{N}(f^n; X, A) = n(f^n; X - A) = |3^n - 1|$$

for all n. Also we have

$$N(f^n; X, A) = 2|3^n - 1|$$

for all n.

EXAMPLE 3.2. Let X be the solid torus in Euclidean 3-space R^3 which is obtained by rotating the 2-disk in the x_1x_3-plane of radius 1 and centered at $(2,0,0)$ about the x_3-axis, and let A be the 2-dimensional torus which bounds X. We consider R^3 as $\mathbf{C} \times R^1$, where $\mathbf{C}$ is the complex plane, and label the points of X as $(re^{i\theta}, t)$, where $re^{i\theta} \in \mathbf{C}$ and $t \in R^1$, with $1 \leq \theta < 2\pi$ and $-1 \leq t \leq 1$. Let $f : (X, A) \to (X, A)$ be the map given by

$$f(re^{i\theta}, t) = (re^{id\theta}, -|t|),$$

where $d \neq 1$ is an integer. As any circle of latitude is a deformation retract of X we have $N(f) = |d-1|$ [1, Ch. VIII, p.107; 5, p.21, Theorem 5.4 and p.33, Example 1], and it follows from [5, p.33, Example 2] that $N(\bar{f}) = |d-1|$. The fixed point set of f lies in $t \leq 0$ and consists of $|d-1|$ half-disks. Each half-disk forms an essential fixed point class of f and contains one essential fixed point class of $\bar{f}$ on its boundary because the arcs of the boundary S^1 of the rotated 2-disk from $(e^{\frac{i2n\pi}{d-1}}, 0)$ to $(3e^{\frac{i2n\pi}{d-1}}, 0)$ for $n = 0, 1, 2, \cdots, d-2$, passing through the south pole show this. Hence

$$N(f; X, A) = |d-1|.$$

$Keri_\pi \simeq \mathbf{Z}$ is generated by the loop α obtained by travelling the boundary S^1 of the 2-disk once, starting $(e^{i0}, 0) = x_0$, in the counter-clockwise direction. Now select the path w_0 to be the constant path at x_0. Then we have

$$\bar{f}^{w_0}(\alpha) = \bar{f}(\alpha) = 0.$$

It is easy to see that $\tilde{N}(f; X, A) = n(f; X - A) = 0$ by Theorem 3.1, Theorem 3.2 and thus, each essential fixed point class of f assumes its index in A.

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