J. Korean Math. Soc. **31** (1994), No. 1, pp. 57–67

ON CONVERGENCE OF FINITE DIFFERENCE SCHEMES FOR GENERALIZED SOLUTIONS OF ELLIPTIC DIFFERENTIAL EQUATIONS

S. K. CHUNG AND H.-J. KIMN

1. Introduction

Let Ω be a rectangular domain in $\mathbb{R}^2$ with boundary $\partial\Omega$. We consider finite difference approximations for the generalized solutions of elliptic differential equations of the form

$$A(x)u(x) = f(x), \quad x \in \Omega, \tag{1.1a}$$

with boundary condition

$$u(x) = 0, \qquad x \in \partial\Omega. \tag{1.1b}$$

Here $A(x)$ is a second order, self-adjoint elliptic operator with smooth coefficients which has the following form

$$A(x)u = -\sum_{l,q=1}^{2} \frac{\partial}{\partial x_l}\Big(a_{lq}(x)\frac{\partial u}{\partial x_q}\Big) + a(x)u.$$

Approximate solutions and error estimates for (1.1) have been obtained through energy arguments using Taylor's Theorem. This traditional approach to the study of the rate of convergence requires a high degree of smoothness for the exact solutions. Cosequently, Taylor's Theorem is not the natural framework in which to establish orders of convergence in weaker norms for finite difference approximations of non-smooth solutions.

Received September 21, 1992.
The first author is supported by KOSEF under Grants 921–0100–011–1 and 91–08–00–01. The second author is supported by KOSEF under Grant 91–08–00–01.

In a celebrated paper on exact finite difference approximations for the generalized solutions of two point boundary value problems, Tikhonov and Samarskii[10], have obtained rates of convergence which are compatible with the smoothness of their solutions. But their methods do not apply to the multidimensional case. Recently, there have been some results on the approximation of the generalized solutions for linear prabolic and hyperbolic partial differential equations via finite difference schemes (see Lazarov *et al.* [7]–[8], Jovanović *et al.* [5]–[6], and Pani *et al.* [9]).

The key to the above approaches is to compare the exact solution with a suitable mollified approximation such as the average values on cells around grid points (instead of point estimates). Such averaging can be defined using the Steklov mollifier. The resulting comparisons of the exact and approximate solutions with the mollified approximation yield sharp orders of convergence in an elegant manner. Similar procedures are utilized here.

In the present paper, we investigate rates of convergence of finite difference schemes for the approximate solution of (1.1). As for the finite element method, we obtain orders of convergence compatible with the smoothness of the solution. A discrete projection technique is then introduced to reduce the regularity to that of the generalized solution.

The preliminary material is given in Section 2. In Section 3, a discrete scheme for (1.1) is analyzed, and stability results for a modified scheme are derived, which yield the required error estimates in the discrete L^2 and H^1 norms. Nitche's technique is applied to the discrete scheme, and the error estimates with reduced regularity on u are derived, which yield $O(h^\alpha)$, $1 < \alpha \leq 2$, convergence in the discrete L^2-norm.

2. Preliminaries

We may assume, without loss of generality, that the domain Ω is the unit square in $\mathbb{R}^2$. For the numerical solution of (1.1), we select a mesh of width $h = \frac{1}{M}$, where M is a positive integer, and cover $\bar{\Omega} = \Omega \cup \partial\Omega$ with a square grid of mesh points $x_{ij} = (ih, jh)$, for $i, j = 0, 1, \ldots, M$. Let $\Omega_h = \{x_{ij} : x_{ij} \in \Omega\}$ and $\partial\Omega_h = \{x_{ij} : x_{ij} \in \partial\Omega\}$. We can cover the whole of $\mathbb{R}^2$ with such a square grid, and will denote it by $\mathbb{R}^2_h$.

For any function v defined on $\mathbb{R}^2_h$, we adopt the following notation: for $x \in \mathbb{R}^2_h$ and $l = 1, 2$,

$$v^{\pm l}(x) = v(x \pm h\mathbf{e}_l), \quad v^{+l,-q}(x) = v(x + h\mathbf{e}_l - h\mathbf{e}_q),$$

and

$$\nabla_l v(x) = \frac{v(x + h\mathbf{e}_l) - v(x)}{h}, \quad \bar{\nabla}_l v(x) = \frac{v(x) - v(x - h\mathbf{e}_l)}{h},$$

where $\mathbf{e}_l$ is the l-th unit vector in $\mathbb{R}^2$.

The Steklov mollifiers are defined in the following manner:

$$S = S_1^2 S_2^2 \text{ with } S_l^2 = S_l^+ S_l^-, \qquad l = 1, 2,$$

where

$$S_l^+ \phi(x) = \int_0^1 \phi(x + sh\mathbf{e}_l)\, ds, \quad S_l^- \phi(x) = \int_{-1}^0 \phi(x + sh\mathbf{e}_l)\, ds.$$

The operators $S_l^{\pm}$ commute, and the following relations hold

$$S_l^2 \phi(x) = \int_{-1}^0 (1 + s)\phi(x + sh\mathbf{e}_l)\, ds + \int_0^1 (1 - s)\phi(x + sh\mathbf{e}_l)\, ds,$$

and

$$(2.1) \qquad S_l^+ \frac{\partial \phi}{\partial x_l} = \nabla_l \phi, \quad S_l^- \frac{\partial \phi}{\partial x_l} = \bar{\nabla}_l \phi.$$

Let $\mathcal{D}_h$ denote the mesh functions defined on $\mathbb{R}^2_h$ which vanish outside of Ω_h. For $u, v \in \mathcal{D}_h$, we now introduce the discrete L^2 space, denoted by $L^2_h(\Omega_h)$, with inner product and norm given by

$$\langle w, v \rangle = h^2 \sum_{x \in \mathbb{R}^2_h} w(x) v(x),$$

and

$$\|w\|_{0,h} = \langle w, w \rangle^{\frac{1}{2}},$$

respectively. Further, let $H_h^1 = H_h^1(\Omega_h)$ denote the discrete analogue of the H^1–Sobolev space with norm $\|w\|_{1,h}^2 = \|w\|_{0,h}^2 + \sum_{l=1}^2 \|\nabla_l w\|^2$. We also introduce a discerete H^2–Sobolev space with norm

$$\|w\|_{2,h}^2 = \|w\|_{1,h}^2 + \sum_{l,q=1}^{2} \|\nabla_l \bar{\nabla}_q w\|_{0,h}^2, \quad w \in \mathcal{D}_h$$

and denote it by $H_h^2 = H_h^2(\Omega_h)$.

Whenever there is no confusion, we write $\|w\|$ and $\|w\|_j$, for $j = 1, 2$, in place of $\|w\|_{0,h}$ and $\|w\|_{j,h}$, respectively. Throughout the paper, $\|\cdot\|_{L^2}$ and $\|\cdot\|_{W^{m,p}(\Omega)}$ will denote the norm in the L^2 and the Sobolev space $W^{m,p}(\Omega)$, respectively. Further, $|\cdot|_{W^{m,p}(\Omega)}$ denotes the seminorm on $W^{m,p}(\Omega)$. In particular, for $p = 2$, we denote $W^{m,p}(\Omega)$ by $H^m(\Omega)$.

For functions v and w which vanish on $\partial\Omega_h$, the following identities are easy consequences of summation by parts:

$$\langle \nabla_l v, w\rangle = -\langle v, \bar{\nabla}_l w\rangle, \qquad l = 1, 2. \tag{2.2}$$

The basic lemma, which will be used in the following sections, are given below. Along with the Bramble-Hilbert Lemma(see, Bramble and Hilbert[1] and Dupont and Scott[4]), the following bilinear version of it will be needed for our convergence analysis. For a proof, we refer the reader to Ciarlet[2].

LEMMA 2.1. *Let $P_{[r]}$ be the set of all polynomials of degree $\leq [r]$, where $[r]$ denotes the largest integer less than $r > 0$. If η is a bounded bilinear functional on $W^{\alpha,p}(\Omega) \times W^{\beta,q}(\Omega)$, with $\alpha, \beta \in (0, \infty)$ and $p, q \in [1, \infty]$ such that*

$$\eta(U, v) = 0, \qquad \forall U \in P_{[\alpha]}(\Omega), \quad \forall v \in W^{\beta,q}(\Omega),$$

$$\eta(u, V) = 0, \qquad \forall u \in W^{\alpha,p}(\Omega), \quad \forall V \in P_{[\beta]}(\Omega),$$

then there exists a positive constant C such that

$$|\, \eta(u, v)\, | \leq C \mid u\, |_{W^{\alpha,p}(\Omega)} | \, v\, |_{W^{\beta,q}(\Omega)},$$
$$\forall u \in W^{\alpha,p}(\Omega), \quad \forall v \in W^{\beta,q}(\Omega).$$

We shall frequently use the following inequality

$$ab \leq \varepsilon a^2 + \frac{1}{4\varepsilon} b^2, \qquad a, b \in R, \quad \varepsilon > 0. \tag{2.3}$$

We write C as a generic positive constant independent of the discretizing parameters h.

3. Discrete Schemes

In this section, the stability and error analysis for a discrete scheme of (1.1) will be given. Let $A_h(x)$ be the finite difference approximations for the operators $A(x)$, defined by

$$A_h(x)V = -\frac{1}{2} \sum_{l,q=1}^{2} \left[\nabla_l\big(a_{lq}(x)\bar{\nabla}_q V\big) + \bar{\nabla}_l\big(a_{lq}(x)\nabla_q V\big)\right] + S(a(x))V,$$

for $x \in \Omega_h$. In order to minimize the regularity imposed on $a(x)$ in the subsequent analysis, it is necessary to work with the Steklov mollification $S(a(x))$ in the above approximations.

Our discrete approximation U on Ω_h to (1.1) is now defined by

$$A_h(x)U(x) = Sf(x), \qquad x \in \Omega_h, \tag{3.1}$$

$$U(x) = 0, \qquad x \in \partial\Omega_h.$$

We first consider a modified difference scheme of (3.1); namely,

$$A_h(x)U(x) = Sf(x) + \sum_{l=1}^{2} \bar{\nabla}_l F(x), \tag{3.2}$$

where F, defined on Ω_h, vanishes on $\partial\Omega_h$.

Let $H^1_{0,h} = \{v \in H^1_h : v = 0 \,\text{on}\, \partial\Omega_h\}$. On the basis of the assumptions imposed on $A(x)$, the following lemma can be verified using summation by parts.

LEMMA 3.1. *For $V, W \in H^1_{0,h}$, we have the following inequalities:*

(1) *the discrete Poincáre inequality:* $\|V\|^2 \le C\sum_{l=1}^{2}\|\nabla_l V\|^2$,
(2) $\langle A_h(t)V, V\rangle \ge c_0\|V\|_1^2$,

where c_0 is a positive constant.

The stability analysis will be formulated in the discrete norm.

THEOREM 3.1. *Let U be a solution of (3.2). Then there exists a constant C such that*

$$\|U\| \le C\big[\|Sf\| + \|F\|\big].$$

Proof. Forming the discrete L^2 inner product between (3.2) and U, we obtain

$$\langle A_h(x)U, U\rangle = \langle Sf, U\rangle + \sum_{l=1}^{2}\langle \bar{\nabla}_l F, U\rangle.$$

Using Lemma 3.1, we obtain

$$\|U\|_1^2 \le C\big[\|Sf\|\|U\| + \|F\|\|U\|_1\big].$$

An application of the inequality (2.3) now completes the proof. □

Using Theorem 3.1, we now derive the following error estimate in $e(x) = u(x) - U(x)$.

THEOREM 3.2. *Let u and U be the solution of (1.1) and (3.1), respectively. Then the error $e(x) = u(x) - U(x)$ satisfies the following estimate*

$$\|e(x)\|_1 \le Ch^{\alpha-1}, \qquad 1 < \alpha \le 3.$$

Proof. From (1.1) and (3.1), it follows that

$$A_h(x)e = \big[A_h(x)u - SA(x)u\big] = G_1(x).$$

Following Jovanović *et al.* [6], the integrand $G_1(x)$ is rewritten as

$$G_1(x) = \sum_{l,q=1}^{2} \bar{\nabla}_l \eta_{lq}(x) + \eta(x),$$

where

$$\eta_{lq} = \eta_{lq}^{(1)} + \eta_{lq}^{(2)} + \eta_{lq}^{(3)} + \eta_{lq}^{(4)}$$

with

$$\eta_{lq}^{(1)} = S_l^+ S_{3-l}^2\Big(a_{lq}\frac{\partial u}{\partial x_q}\Big) - \big(S_l^+ S_{3-l}^2 a_{lq}\big)\Big(S_l^+ S_{3-l}^2\frac{\partial u}{\partial x_q}\Big),$$

$$\eta_{lq}^{(2)} = \Big[S_l^+ S_{3-l}^2 a_{lq} - \frac{1}{2}\big(a_{lq} + a_{lq}^{+l}\big)\Big]\Big(S_l^+ S_{3-l}^2\frac{\partial u}{\partial x_q}\Big),$$

$$\eta_{lq}^{(3)} = \frac{1}{2}\big(a_{lq} + a_{lq}^{+l}\big)\Big[S_l^+ S_{3-l}^2\frac{\partial u}{\partial x_q} - \frac{1}{2}\big(\nabla_q u + \bar{\nabla}_q u^{+l}\big)\Big],$$

$$\eta_{lq}^{(4)} = -\frac{1}{4}\big(a_{lq} - a_{lq}^{+l}\big)\big(\nabla_q u - \bar{\nabla}_q u^{+l}\big),$$

and

$$\eta = (Sa)u - S(au).$$

To estimate η, we first note that

$$\eta = (Sa)(u - Su) + (Sa)(Su) - S(au).$$

As in Pani *et al.*[9], the Bramble-Hilbert Lemma along with Lemma 2.1 yields

$$\|\eta(s)\| \leq C\big(\|a\|_{L^\infty(W^{\alpha-1,\infty})}\big)h^\alpha\|u(x)\|_{H^\alpha(\Omega)}, \quad 1 < \alpha \leq 2.$$

And we obtain the following estimate for η_{lq} as in Jovanović *et al.* [6]

$$\sum_{l,q=1}^{2}\|\eta_{lq}(x)\|^2 \leq Ch^{\alpha-1}\|u(x)\|_{H^\alpha(\Omega)}, \quad 1 < \alpha \leq 3,$$

where the constant C depends on $\max_{l,q}\|a_{lq}\|_{L^\infty(W^{\alpha-1,\infty})}$. Theorem 3.1 now completes the proof. □

We now derive the L^2-error estimates with order of convergence compatible with the regularity on the generalized solution u with reduced regularity.

THEOREM 3.3. *There exists a constant C, which depends on u, such that*

$$\|e\| + h\|e\|_1 \leq Ch^\alpha, \quad 1 < \alpha \leq 2.$$

Proof. For the estimation of $\|e\|_1$, we take the discrete inner product of (3.1) with e and obtain

$$\langle A_h e, e\rangle = \langle A_h u - SAu, e\rangle = \langle G_1, e\rangle.$$

Using the estimate for G_1 from Theorem 3.2, we obtain

$$\left|\langle G_1, e\rangle\right| \leq Ch^{\alpha-1}\|e\|_{H^\alpha}, \quad 1 < \alpha \leq 2.$$

Here the constant C depends on $\|u\|_{H^2(\Omega)}$, $\max_{l,q} \mid a_{lq} \mid_{W^{1,\infty}}$, and $|a|_{W^{1,\infty}}$. Now from the coercivity of the operator A_h, the required estimate follows for $\|e\|_1$.

For the discerete L^2 estimate, we define Φ as the solution of

$$\begin{aligned} A_h\Phi &= e, \qquad x \in \Omega_h, \\ \Phi &= 0, \qquad x \in \partial\Omega_h. \end{aligned} \tag{3.3}$$

Because of the coercivity of A_h, Φ is a unique solution of (3.3) with appropriate regularity

$$\|\Phi\|_2 \leq C\|e\|. \tag{3.4}$$

Forming the discrete inner product between (3.3) and e, we find that

$$\begin{aligned} \langle e, e\rangle = \langle A_h e, \Phi\rangle &= \langle A_h u - SAu, \Phi\rangle \\ &= \langle G_1, \Phi\rangle. \end{aligned} \tag{3.5}$$

For the estimation of G_1, we decompose G_1, as in Theorem 3.2, and obtain, with an application of Lemma 2.1,

$$\left|\sum_{l,q=1}^{2}\langle \eta_{lq}^{(1)}, \nabla_l\Phi\rangle\right| \leq Ch^\alpha\left(\sum_{l,q=1}^{2}|a_{lq}|_{W^{1,\infty}(\Omega)}\right)|u|_{H^\alpha(\Omega)}\|\Phi\|_1,\ 1 < \alpha \leq 2.$$

For the estimation of $\sum_{l,q=1}^{2}\langle\eta_{lq}^{(3)},\nabla_l\Phi\rangle$, we consider it term by term.
For $l=1, q=2$,

$$\begin{aligned}\langle\eta_{12}^{(3)},\nabla_1\Phi\rangle &= \left\langle S_1^+S_2^2\frac{\partial u}{\partial x_2}-\frac{1}{2}(\nabla_2 u+\bar{\nabla}_2 u^+),\frac{1}{2}(a_{12}+a_{12}^+)\nabla_1\Phi\right\rangle \\ &= \left\langle \nabla_2\left[S_1^+S_2^- u-\frac{1}{2}(u+u^{+1,-2})\right],\frac{1}{2}(a_{12}+a_{12}^{+1})\nabla_1\Phi\right\rangle.\end{aligned}$$

Here, we have used (2.1) and $\bar{\nabla}_l u=\nabla_l u(x-e_l h)$. The application of (2.2) yields

$$\begin{aligned}|\langle\eta_{12}^{(3)},\nabla_1\Phi\rangle| &= \left\langle S_1^+S_2^- u-\frac{1}{2}(u+u^{+1,-2}),\bar{\nabla}_2\Big(\frac{1}{2}(a_{12}+a_{12}^{+1})\nabla_1\Phi\Big)\right\rangle \\ &\le C\|S_1^+S_2^- u-\frac{1}{2}(u+u^{+1,-2})\|\|\Phi\|_2.\end{aligned}$$

Since $S_1^+S_2^- u-\frac{1}{2}(u+u^{+1,-2})$ vanishes for all $u\in P_1$, an application of the Bramble-Hilbert Lemma yields

$$\left\|S_1^+S_2^- u-\frac{1}{2}(u+u^{+1,-2})\right\|\le Ch^\alpha|u|_{H^\alpha(\Omega)}.$$

The estimator for $\langle\eta_{21}^{(3)},\nabla_1\Phi\rangle$ is obtained in a similar manner.
For $l=1, q=1$,

$$\begin{aligned}\langle\eta_{11}^{(3)},\nabla_1\Phi\rangle &= \left\langle S_1^+S_2^2\frac{\partial u}{\partial x_1}-\frac{1}{2}(\nabla_1 u+\bar{\nabla}_1 u^{+1}),\frac{1}{2}(a_{11}+a_{11}^{+1})\nabla_1\Phi\right\rangle \\ &= \left\langle \nabla_1(S_2^2 u-u),\frac{1}{2}(a_{11}+a_{11}^{+1})\nabla_1\Phi\right\rangle \\ &= -\left\langle (S_2^2 u-u),\frac{1}{2}\bar{\nabla}_1((a_{11}+a_{11}^{+1})\nabla_1\Phi)\right\rangle.\end{aligned}$$

Since S_2^2u-u vanishes for all $u\in P_1$, an application of the Bramble-Hilbert Lemma again yields

$$\left|\langle\eta_{11}^{(3)},\nabla_1\Phi\rangle\right|\le Ch^\alpha|u|_{H^\alpha(\Omega)}\|\Phi\|_2,\quad 1<\alpha\le 2.$$

The estimate for $\langle \eta_{22}^{(3)}, \nabla_2 \Phi \rangle$ is obtained in a similar manner.

Using Lemma 2.1, we can derive the following estimates

$$\left| \sum_{l,q=1}^{2} \langle \eta_{lq}^{(2)}, \nabla_l \Phi \rangle \right| \le Ch^2 \sum_{l,q=1}^{2} |a_{lq}|_{W^{\alpha,\infty}} \|u\|_{H^{\alpha-1}(\Omega)} \|\Phi\|_2,$$

$$\left| \sum_{l,q=1}^{2} \langle \eta_{lq}^{(4)}, \nabla_l \Phi \rangle \right| \le Ch^2 (\sum_{l,q=1}^{2} |a_{lq}|_{W^{\alpha-1,\infty}}) |u|_{H^{\alpha}(\Omega)} \|\Phi\|_2,$$

and from the Bramble–Hilbert lemma we obtain

$$\begin{aligned} |\langle \eta, \Phi \rangle| &\le \left|\langle (Sa)(u - Su) + (Sa)(Su) - S(au), \Phi \rangle\right| \\ &\le Ch^\alpha \big[|a|_{L^\infty(\Omega)} |u|_{H^\alpha(\Omega)} + |a|_{W^{\alpha-1,\infty}(\Omega)} |u|_{H^{\alpha-1}(\Omega)}) \|\Phi\| \big], \\ &\quad 1 < \alpha \le 2. \end{aligned}$$

It therefore follows that

$$|\langle G_2, \Phi \rangle| \le Ch^\alpha \|\Phi\|_2.$$

Using the regularity condition (3.4), we obtain

$$\|e\| \le Ch^\alpha \|u\|_{H^\alpha(\Omega)}, \quad 1 < \alpha \le 2,$$

with C depending on $|a|_{W^{\alpha-1,\infty}(\Omega)}$ and $\max_{1 \le l,q \le 2} \|a_{lq}\|_{W^{\alpha-1,\infty}(\Omega)}$.
This completes the proof of Theorem 3.3. □

References

1. J. H. Bramle and S. R. Hilbert, *Bounds for a class of linear functionals with applications to Hermite interpolation*, Numer. Math. **16** (1971), 362–369.
2. P. G. Ciarlet, *The finite element method for elliptic problems*, North–Holand, Amsterdam, 1978.
3. E. G. D'jakonov, *On the stability of differnce schemes for some nonstationary problems*, Topics in Numerical Analysis, edited by J. J. H. Miller, Academic Press, 1972, pp. 63–87.
4. T. Dupont and R. Scott, *Polynomial approximations of functions in Sovolev spaces*, Math. Comp. **34** (1980), 441–463.

5. B. S. Jovanović, L. D. Ivanović and E. E. Süli, *Convergence of finite-difference for elliptic equations with variable coefficients*, IMA J. of Numer. Anal. **7** (1987), 301–305.
6. B. S. Jovanović, L. D. Ivanović and E. E. Süli, *Convergence of a finite-difference scheme for second-order hyperbolic equations with variable coeffcients*, IMA J. of Numer. Anal. **7** (1987), 39–45.
7. R. D. Lazarov, V. L. Makarov and A. A. Samarskii, *Application of exact difference schemes to the construction and study of difference schemes for generalized solutions*, Math. USSR Sbornik **45** (1983), 461–471.
8. R. Lazarov, V. L. Makarov and W. Weinelt, *On the convergence of difference schemes for the approximation of solutions* $u \in W_2^m$ $(m > 0.5)$ *of elliptic equations with mixed derivatives*, Numer. Math. **44** (1984), 223–232.
9. A. K. Pani, S. K. Chung and R. S. Anderssen, *On convergence of finite difference schemes for generalized solutions of parabolic and hyperbolic partial differential equations*, CMA Report CMA–MR1–91, The Australian National University, Canberra, Australia (1991).
10. A. N. Tikhonov and A. A. Samarskiĭ, *Homogeneous differnce schemes*, USSR Math. Comp. **1** (1961), 5–67.

Department of Mathematics Education
Seoul National University
Seoul 151–742, Korea

Department of Computer Science
Ajou University
Suwon 441–749, Korea