J. Korean Math. Soc. **31** (1994), No. 1, pp. 29–48

A GENERALIZATION OF THE CARISTI–KIRK FIXED POINT THEOREM AND ITS APPLICATIONS TO MAPPING THEOREMS

J.S. BAE, E.W. CHO AND S.H. YEOM

0. Introduction

Let P be a nonlinear operator mapping a Banach space X into a Banach space Y. There are many approches to the study of solvability of the equation $Px = y$ for $y \in Y$. One of them is a mapping theoretic method in order to obtain precise ranges of operators. Also we are interested in prediction, by looking at local or infinitesimal assumptions on the operator P, whether P is a homeomorphism, is surjective, or has fixed points.

The original investigation for these works came from the fixed points. Fixed point theorems having local hypotheses have been widely studied, including local contractions, local expansions and differentiable mappings. Also fixed point theorems are good tools for our purposes.

The purpose in this paper is to study several mapping theorems. The idea of our method is based on the Caristi-Kirk fixed point theorem [11], which is also equivalent to the minimization theorem of Ekelend [13,14]. In [8], Brezis and Browder gave an ordering principle which can be applied to prove the Caristi-Kirk fixed point theorem [11]. Our main point of view is to extend the Brezis-Browder ordering principle to a more suitable form, and we apply it to obtain our mapping theorems. In fact, many authors, Cramer and Ray [12], Kirk and Caristi [15], Ray [19], Ray and Walker [20], and Rosenholtz and Ray [21] proved their mapping theorems by using the Caristi-Kirk fixed point theorem or the idea contained in the Brezis-Browder ordering principle. Our ordering principle enables us to extend the above mentioned authors' results to more general settings.

Received August 25, 1992.

The present studies were supported by the Basic Science Research Institute Program, Ministry of Education, 1991-1992, Project No. BSRI-91-118.

We investigate an ordering principle in Section 1 and in Section 2, we apply it to obtain several fixed point theorems in complete metric spaces.

Section 3 deals with Altman's weak contractor directions and weak directional contractions, and we give theorems concerning the solvability of the equation $Px = y$. In Section 4, we will give mapping theorems for Gateaux differentiable operators and we estimate the precise range of the given operator.

§1. An ordering principle

The basic technique for our fixed point theorems and mapping theorems is based on the following theorem which is similar to the Brezis-Browder ordering principle [8].

THEOREM 1. *Let $\triangleleft$ be a reflexive relation on a nonempty set M and $\phi : M \to R$ a function bounded from below, which satisfies the following two conditions ;*

(1) if $x \triangleleft y$ and $x \neq y$, then $\phi(x) > \phi(y)$; and

(2) for any sequence $\{x_n\}$ with $x_n \triangleleft x_{n+1}$ for $n = 1, 2, \cdots$, there is a $z \in M$ such that for any positive integer k, there is a positive integer $m \geq k$ with $x_m \triangleleft z$.

Then for any given $x_0 \in M$, there exists a $z \in M$ such that

(3) there is a finite number of elements $x_1, \cdots, x_n \in M$ satisfying $x_{i-1} \triangleleft x_i$ for $1 \leq i \leq n$ and $x_n \triangleleft z$; and

(4) $z \triangleleft x$ implies $z = x$.

Proof. For any $x \in M$, we put

$$\begin{aligned} S(x) = \{y \in M |\ & \text{there are finite elements } x_1, \cdots, x_n \in M \\ & \text{such that } x \triangleleft x_1, x_n \triangleleft y \text{ and } x_{i-1} \triangleleft x_i \text{ for } 2 \leq i \leq n\}. \end{aligned}$$

Then we know that if $y \in S(x)$, then $S(y) \subseteq S(x)$. Starting at the given point x_0, we can obtain a sequence $\{x_n\}$ in M with $x_n \in S(x_{n-1})$ and

$$\phi(x_n) \leq \inf\{\phi(y) | y \in S(x_{n-1})\} + \frac{1}{n}$$

for all $n \geq 1$ by using induction, since ϕ is bounded from below. Then the condition (2) yields that there is an element $z \in M$ with $z \in S(x_n)$

for all $n \geq 0$. Therefore by (1) we know that the sequence $\{\phi(x_n)\}$ is nonincreasing and $\phi(z) \leq \phi(x_n) \leq \phi(z) + \frac{1}{n}$ for all $n \geq 1$, and hence $\lim \phi(x_n) = \phi(z)$. Now suppose that $z \triangleleft y$ for some $y \in M$. Then clearly we have $y \in S(x_n)$ for all $n \geq 0$, so that

$$\phi(y) \leq \phi(x_n) \leq \phi(y) + \frac{1}{n}, \quad n = 1, 2, \cdots .$$

Therefore we have $\phi(y) = \lim \phi(x_n) = \phi(z)$, and hence by (1) we see that $y = z$. Also since $z \in S(x_0)$, (3) holds trivially by the definition of $S(x_0)$. This completes the proof.

Theorem 1 is different from Lemma 3.1 of [5] since Lemma 3.1 of [5] does not state the conclusion (3). Also we remark that in Theorem 1, the relation $\triangleleft$ need not be an order relation since it is not assumed to be transitive. However, this relation has transitivity implicitly. We can obtain the Brezis-Browder's result as a direct application of Theorem 1.

COROLLARY 2. *Let $\leq$ be an order relation on a nonempty set M and $\phi : M \to R$ a function bounded from below, which satisfies the following two conditions ;*

(1)′ *if $x \leq y$ and $x \neq y$, then $\phi(x) > \phi(y)$; and*

(2)′ *any nondecreasing sequence in M has an upper bound in M.*

Then for any given $x_0 \in M$, there is a maximal element $z \in M$ with $x_0 \leq z$.

§2. Fixed point theorems

We recall that a function $\phi : X \to R$ is *upper semicontinuous* (u.s.c.), where X is a topological space if for any $t \in R$, the set $\{x \in X | \phi(x) < t\}$ is open in X. Also $\phi : X \to R$ is said to be *lower semicontinuous* (l.s.c) if for any $t \in R$, the set $\{x \in X | \phi(x) > t\}$ is open in X. We note that if X is a metric space, then $\phi : X \to R$ is u.s.c (resp. l.s.c.) if and only if for any sequence $\{x_n\}$ in X with $x_n \to x$,

$$\limsup \phi(x_n) \leq \phi(x) \quad \big(\text{resp. } \liminf \phi(x_n) \geq \phi(x)\big).$$

By applying Theorem 1, we first obtain the following fixed point theorem.

THEOREM 3. *Let X be a complete metric space, $\phi : X \to [0,\infty)$ a l.s.c. function and $c : [0,\infty) \to [0,\infty)$ an u.s.c. function. Let $g : X \to X$ be a mapping such that for each $x \in X$,*

$$d(x, gx) \leq \max\{c(\phi(x)), c(\phi(gx))\}\{\phi(x) - \phi(gx)\} \tag{5}$$

holds. Then g has a fixed point in X.

Proof. Define a relation $\triangleleft$ on X such that for $x, y \in X$,

$$x \triangleleft y \text{ iff } d(x,y) \leq \max\{c(\phi(x)), c(\phi(y))\}\{\phi(x) - \phi(y)\}.$$

Then clearly $\triangleleft$ is reflexive and satisfies the condition (1). Let $\{x_n\}$ be a sequence in X with $x_n \triangleleft x_{n+1}$ for $n = 1, 2, \cdots$. Then the condition (1) implies that $\{\phi(x_n)\}$ is nonincreasing. Therefore, $\lim \phi(x_n) = \alpha(\geq 0)$ exists. Since c is u.s.c., we have $\limsup c(\phi(x_n)) \leq c(\alpha)$. Therefore we can choose a natural number N such that $c(\phi(x_n)) \leq c(\alpha) + 1$ for any $n \geq N$. For $n \geq N$, $x_n \triangleleft x_{n+1}$ implies

$$\begin{aligned} d(x_n, x_{n+1}) &\leq \max\{c(\phi(x_n)), c(\phi(x_{n+1}))\}\{\phi(x_n) - \phi(x_{n+1})\} \\ &\leq \{c(\alpha) + 1\}\{\phi(x_n) - \phi(x_{n+1})\}, \end{aligned}$$

so that for $m > n \geq N$, we have

$$d(x_n, x_m) \leq \{c(\alpha) + 1\}\{\phi(x_n) - \phi(x_m)\}.$$

Therefore we conclude that $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, $\lim x_n = z(\in X)$ exists.

Now we claim that for this z, the condition (2) holds, that is, for any given positive integer k, there is an integer $m \geq k$ such that $x_m \triangleleft z$. We consider possible three cases.

(I) Suppose that there is an integer $m \geq k$ such that

$$c(\phi(x_m)) = \sup\{c(\phi(x_n)) | n \geq m\}.$$

Then for any $n \geq m$, we get

$$d(x_n, x_{n+1}) \leq c(\phi(x_m))\{\phi(x_n) - \phi(x_{n+1})\}.$$

Therefore for $n > m$, by adding both sides of the above inequality through m to $n-1$, we get

$$d(x_n, x_m) \leq c\big(\phi(x_m)\big)\big\{\phi(x_m) - \phi(x_n)\big\}.$$

By letting $n \to \infty$, since $\lim \phi(x_n) \geq \phi(z)$ we have

$$d(x_m, z) \leq c\big(\phi(x_m)\big)\big\{\phi(x_m) - \phi(z)\big\},$$

which means that $x_m \lhd z$.

(II) Suppose that $\lim \phi(x_n) = \alpha = \phi(z)$ and for any integer $m \geq k$, there is an integer $n > m$ such that $c\big(\phi(x_n)\big) > c\big(\phi(x_m)\big)$. Since c is u.s.c., we know that

$$\sup\big\{c(\phi(x_n))|n \geq k\big\} = \limsup c\big(\phi(x_n)\big) \leq c\big(\phi(z)\big).$$

Therefore for $n \geq k$, we get

$$\begin{aligned} d(x_n, x_{n+1}) &\leq \max\big\{c\big(\phi(x_n)\big), c\big(\phi(x_{n+1})\big)\big\}\big\{\phi(x_n) - \phi(x_{n+1})\big\} \\ &\leq c\big(\phi(z)\big)\big\{\phi(x_n) - \phi(x_{n+1})\big\}, \end{aligned}$$

and hence by adding both sides of the above inequality through k to $n-1$, we have

$$d(x_n, x_k) \leq c\big(\phi(z)\big)\big\{\phi(x_k) - \phi(x_n)\big\}.$$

Also by letting $n \to \infty$, we have

$$d(x_k, z) \leq c\big(\phi(z)\big)\big\{\phi(x_k) - \phi(z)\big\},$$

which shows that $x_k \lhd z$.

(III) Suppose that $\lim \phi(x_n) = \alpha > \phi(z)$ and for any integer $m \geq k$, there is an integer $n > m$ such that $c\big(\phi(x_n)\big) > c\big(\phi(x_m)\big)$. Then also we know that

$$\sup\big\{c\big(\phi(x_n)\big)|\ n \geq k\big\} = \limsup c\big(\phi(x_n)\big) = \beta > 0.$$

Therefore we can find an integer $m > k$ such that $c(\phi(x_m)) \geq \frac{1}{2}\beta$ and for $n \geq m$, $\phi(x_n) \leq 2\alpha - \phi(z)$. Then we have for $n \geq m$,

$$\begin{aligned} d(x_n, x_{n+1}) &\leq \max\{c(\phi(x_n)), c(\phi(x_{n+1}))\}\{\phi(x_n) - \phi(x_{n+1})\} \\ &\leq \beta\{\phi(x_n) - \phi(x_{n+1})\} \\ &\leq 2c(\phi(x_m))\{\phi(x_n) - \phi(x_{n+1})\}. \end{aligned}$$

By adding both sides of the above inequality through m to $n-1$ for $n > m$, we have

$$d(x_m, x_n) \leq 2c(\phi(x_m))\{\phi(x_m) - \phi(x_n)\}.$$

By letting $n \to \infty$, we get

$$d(x_m, z) \leq 2c(\phi(x_m))\{\phi(x_m) - \alpha\}.$$

Also since $\phi(x_m) \leq 2\alpha - \phi(z)$, we have

$$\begin{aligned} 2\{\phi(x_m) - \alpha\} &= \phi(x_m) - 2\alpha + \phi(x_m) \\ &\leq \phi(x_m) - \phi(z). \end{aligned}$$

Therefore, finally we obtain

$$d(x_m, z) \leq c(\phi(x_m))\{\phi(x_m) - \phi(z)\},$$

which yields $x_m \lhd z$.

Now by applying Theorem 1, we get $z \in X$ such that $z \lhd y$ implies $z = y$. However, the condition (5) means that $x \lhd g(x)$ for any $x \in X$. Therefore $gz = z$, which completes the proof.

In Theorem 3, if the function c is nonincreasing, then we can have a more stronger result as follows.

THEOREM 4. *Let X be a complete metric space, $\phi : X \to [0, \infty)$ a l.s.c. function, and $c : [0, \infty) \to [0, \infty)$ a nonincreasing function. Let $g : X \to X$ be a mapping such that for each $x \in X$,*

$$d(x, gx) \leq c(\phi(x))\{\phi(x) - \phi(gx)\} \tag{6}$$

holds. Then for any given $x_0 \in X$, g *has a fixed point* $z \in X$ *with*

$$d(x_0, z) \le \int_0^{\phi(x_0)} c(t)dt.$$

Proof. Define a new function $\psi : X \to [0, \infty)$ by

$$\psi(x) = \int_0^{\phi(x)} c(t)dt, \quad \text{for } x \in X.$$

Then we see that ψ is also a l.s.c. function. In fact, if $x_n \to x$ in X, then $\liminf \phi(x_n) \ge \phi(x)$ shows that

$$\liminf \psi(x_n) = \liminf \int_0^{\phi(x_n)} c(t)dt \ge \int_0^{\phi(x)} c(t)dt = \psi(x),$$

so that ψ also l.s.c.. Now define an order relation $\le$ on X by

$$x \le y \text{ iff } d(x, y) \le \psi(x) - \psi(y).$$

Then it can be easily shown that $(1)'$ and $(2)'$ hold (see [7,14,17]). Therefore, by Corollary 2, for the given point $x_0 \in X$, there is a maximal element $z \in X$ with $x_0 \le z$. But by the condition (6), we know that

$$\begin{aligned} d(z, gz) &\le c\big(\phi(z)\big)\big\{\phi(z) - \phi(gz)\big\} \\ &\le \int_{\phi(gz)}^{\phi(z)} c(t)dt = \psi(z) - \psi(gz), \end{aligned}$$

so that $z \le gz$. By the maximality of z we know $gz = z$, and $x_0 \le z$ means that

$$\begin{aligned} d(x_0, z) \le \psi(x_0) - \psi(z) &= \int_{\phi(z)}^{\phi(x_0)} c(t)dt \\ &\le \int_0^{\phi(x_0)} c(t)dt. \end{aligned}$$

This completes the proof.

By the same method as in the proof of Theorem 4, we can have the following theorem.

THEOREM 5. *Let X be a complete metric space, $\phi : X \to [0,\infty)$ a l.s.c. function, and $c : [0,\infty) \to [0,\infty)$ a nondecreasing function. Let $g : X \to X$ be a mapping such that for each $x \in X$,*

$$d(x,gx) \leq c\big(\phi(gx)\big)\big\{\phi(x) - \phi(gx)\big\} \tag{7}$$

holds. Then for any given $x_0 \in X$, g has a fixed point $z \in X$ with

$$d(x_0,z) \leq \int_0^{\phi(x_0)} c(t)dt.$$

Proof. As in the proof of Theorem 4, define $\psi : X \to [0,\infty)$ by

$$\psi(x) = \int_0^{\phi(x)} c(t)dt, \qquad \text{for } x \in X.$$

Then also we know that ψ is a l.s.c. function. Now define an order relation $\leq$ on X by

$$x \leq y \ \text{ iff } \ d(x,y) \leq \psi(x) - \psi(y).$$

According to Corollary 2, for the given point $x_0 \in X$, there exists a maximal element $z \in X$ with $x_0 \leq z$. But by the condition (7), we know that

$$\begin{aligned} d(z,gz) &\leq c\big(\phi(gz)\big)\big\{\phi(z) - \phi(gz)\big\} \\ &\leq \int_{\phi(gz)}^{\phi(z)} c(t)dt, \end{aligned}$$

since c is nondecreasing, so that $z \leq gz$. Therefore the maximality of z implies $gz = z$. Also since $x_0 \leq z$,

$$\begin{aligned} d(x_0,z) \leq \psi(x_0) - \psi(z) &= \int_{\phi(z)}^{\phi(x_0)} c(t)dt \\ &\leq \int_0^{\phi(x_0)} c(t)dt. \end{aligned}$$

This completes the proof.

Note that if $c(t) = 1$, then Theorems 3-5 are just the Caristi-Kirk fixed point theorem. Also our Theorems 4-5 can be compared with Theorem 2.1 of Ray and Walker [20] (cf. [7, 17]).

§3. Weak contractor directions

Altman [1] introduced the concept of contractor directions and directional contractions to apply to the solvability of the equation $Px = y$. Their concepts are appeared in some local assumptions on the operator P such as differentiability.

Let X be an abstract set, Y a Banach space, and let $P : X \to Y$ be a mapping from X to Y. Given an upper semicontinuous function $q : [0, \infty) \to [0, 1)$, we define weak contractor directions for P at x such that $y \in Y$ is said to be a *weak contractor direction* if there exists a positive number $\epsilon = \epsilon(x, y) \leq 1$ and an element $\overline{x} \in X$ satisfying

$$\|P\bar{x} - Px - \epsilon y\| \leq \epsilon q(\|y\|)\|y\|. \tag{8}$$

Also we denote $\Gamma_x^*(P)$ by the set of weak contractor directions for P at x. If $q(\|y\|) = q$ is constant, then $\Gamma_x^*(P)$ is a set of *contractor directions* for P at x, denoted by $\Gamma_x(P)$ (see [1,2,3,4]).

Our first mapping theorem is the following existence result of the solution of the equation $Px = y$.

THEOREM 6. *Let X be a nonempty set , Y a Banach space, and $P : X \to Y$ a mapping such that $P(X)$ is closed in Y. Suppose that $y \in Y$ is fixed and $y - Px \in \Gamma_x^*(P)$ for each $x \in X$. Then the equation $Px = y$ has a solution in X.*

Proof. Putting $M = y - P(X) = \{y - Px | x \in X\}$, we know that M is a complete metric space, since $P(X)$ is closed. Since for any $x \in X, y - Px \in \Gamma_x^*(P)$, by (8) there is an $\bar{x} \in X$ such that

$$\big\|P\bar{x} - Px - \epsilon(y - Px)\big\| \leq \epsilon q\big(\|y - Px\|\big)\|y - Px\|. \tag{9}$$

for $0 < \epsilon \leq 1$ and an u.s.c. function $q : [0, \infty) \to [0, 1)$. From (9) we have

$$\big\|(y - Px) - (y - P\bar{x})\big\| \leq \epsilon\big\{1 + q(\|y - Px\|)\big\}\|y - Px\|. \tag{10}$$

Also from (9) we obtain

$$\|y - P\bar{x}\| - (1 - \epsilon)\|y - Px\| \leq \epsilon q\big(\|y - Px\|\big)\|y - Px\|,$$

so that we get

$$(11)\qquad \epsilon\{1-q(\|y-Px\|)\}\|y-Px\| \le \|y-Px\| - \|y-P\bar{x}\|.$$

By combining (10) and (11). we finally have

$$(12)\ \|(y-Px)-(y-P\bar{x})\| \le \frac{1+q(\|y-Px\|)}{1-q(\|y-Px\|)}(\|y-Px\|-\|\|y-P\bar{x}\|).$$

Now let $\phi : M \to [0,\infty)$ be $\phi(x)=\|x\|$, and let $c:[0,\infty)\to[0,\infty)$ be $c(t)=\{1+q(t)\}/\{1-q(t)\}$. Then c is also u.s.c., since so is q. Now by putting $g(y-Px)=y-P\bar{x}$, we have a mapping $g : M \to M$ satisfying the condition (5) as a result of (12).

Therefore by Theorem 3, g has a fixed point in M, that is, there is a point $x_0 \in X$ such that $g(y-Px_0)=y-Px_0$. But this means that $P\bar{x}_0 = Px_0$, and hence by (9) we have

$$\epsilon\|y-Px_0\| \le \epsilon q(\|y-Px_0\|)\|y-Px_0\|.$$

Also since $0 \le q(\|y-Px_0\|) < 1$, we conclude that $Px_0 - y = 0$, which completes the proof.

Note that if $y=0$, then Theorem 6 is a consequence of Altman's result [4]. As a direct application of Theorem 6 we have the following surjectivity result.

COROLLARY 7. *Let X be a nonempty set, Y a Banach space, and let $P : X \to Y$ be a mapping such that $P(X)$ is closed in Y. Suppose that for any $x \in X$, $\Gamma_x^*(P) = Y$. Then P is surjective, that is, $P(X)=Y$.*

Proof. Since $y - Px \in \Gamma_x^*(P)$ for any $x \in X$ and any $y \in Y$, by Theorem 6, the equation $Px = y$ has a solution for all $y \in Y$. Therefore P is surjective.

If the closedness of $P(X)$ is omitted, then we need more stronger conditions to obtain mapping theorems for nonlinear operators of contractor directional type.

We say that an operator P from a subset D of a metric space X into a metric space Y has closed graph if its graph is closed in $D \times Y$, that is, for any sequence $\{x_n\} \subseteq D$ with $x_n \to x \in D$ and $Px_n \to y, Px = y$

holds (see [4] for the definition). Also we say that P has *closed graph* in $X \times Y$ if its graph is closed in $X \times Y$, that is, for any sequence $\{x_n\} \subseteq D$ with $x_n \to x$ and $Px_n \to y$, it follows that $x \in D$ and $Px = y$ (see [1,2,3] for the definition). Note that if P has closed graph in $X \times Y$, then it has closed graph, and if D is closed in X, then the converse holds.

For a metric space X and $x \in X$, we denote by $B(x;r)$ the set

$$\{y \in X | d(x,y) < r\},$$

and $\overline{B}(x;r)$ its closure for $r > 0$. Also conveniently, we set $B(x;\infty) = X$.

THEOREM 8. *Let D be a nonempty subset of a complete metric space X, Y a Banach space, and let $P : D \to Y$ be a mapping having closed graph in $X \times Y$. Suppose that there are a constant $q \in (0,1)$ and an u.s.c. function $c : [0,\infty) \to [\alpha,\infty)$ with $\alpha > 0$ such that for any $x \in D$, there are an $\overline{x} \in D$ and $0 < \epsilon = \epsilon(x) \le 1$ satisfying*

$$\|P\overline{x} - (1-\epsilon)Px\| \le q\epsilon\|Px\| \tag{13}$$

and

$$d(x,\overline{x}) \le \epsilon c(\|Px\|)\|Px\|. \tag{14}$$

Then the equation $Px = 0$ has a solution in D. Moreover, if c is nonincreasing, then for any given point $x_0 \in D$, the equation $Px = 0$ has a solution in $D \cap \overline{B}(x_0;r)$, where

$$r = (1-q)^{-1} \int_0^{\|Px_0\|} c(t)dt.$$

Also as noted above, if D is closed in X, then we can simply assume that P has closed graph.

Proof. Give a new metric ρ on D by setting

$$\rho(x,y) = \max\{d(x,y), (1+q)^{-1}\alpha\|Px - Py\|\}, \ x,y \in D.$$

Then we can prove that the metric space (D, ρ) is complete. In fact, suppose that $\{x_n\}$ is a Cauchy sequence in (D, ρ). Then it can be easily shown that $\{x_n\}$ and $\{Px_n\}$ are Cauchy sequences in X and Y, respectively. Therefore $x_n \to x$ and $Px_n \to y$ for some $x \in X$ and $y \in Y$, since X and Y are both complete. Since P has closed graph in $X \times Y$, it follows that $x \in D$ and $Px = y$. Then clearly $\rho(x_n, x) \to 0$ as $n \to \infty$. Therefore (D, ρ) is complete.

Now also set $\phi : D \to [0, \infty)$ by $\phi(x) = \|Px\|$ for $x \in D$. Then ϕ is continuous with respect to the new metric ρ on D.

For each $x \in D$ we can find $\bar{x} \in D$ satisfying (13) and (14). Now from (13) we have

$$\|Px - P\bar{x}\| \leq \epsilon(1+q)\|Px\|. \tag{15}$$

Also from (13) we get

$$\epsilon(1-q)\|Px\| \leq \|Px\| - \|P\bar{x}\|. \tag{16}$$

By combining (15) and (16) we obtain

$$\|Px - P\bar{x}\| \leq \frac{1+q}{1-q}\big(\|Px\| - \|P\bar{x}\|\big). \tag{17}$$

And by (14) and (16) we have

$$d(x, \bar{x}) \leq \frac{1}{1-q} c\big(\|Px\|\big)\big(\|Px\| - \|P\bar{x}\|\big). \tag{18}$$

By putting $gx = \bar{x}$, we see that the mapping $g : D \to D$ satisfies

$$\rho(x, gx) \leq \frac{1}{1-q} c\big(\phi(x)\big)\big\{\phi(x) - \phi(gx)\big\}, \ x \in D.$$

Therefore by applying Theorem 3, g has a fixed point $z \in D$, that is, $gz = \bar{z} = z$. Since $0 < q < 1$, by (13) we can conclude that $Pz = 0$. Moreover, suppose that c is nonincreasing. Then by applying Theorem 4, for any given $x_0 \in X, g$ must have a fixed point $z \in D$ with

$$\begin{aligned} d(x_0, z) \leq \rho(x_0, z) &\leq \int_0^{\phi(x_0)} \frac{1}{1-q} c(t)dt \\ &= \frac{1}{1-q} \int_0^{\|Px_0\|} c(t)dt. \end{aligned}$$

Since this z should be a solution of the equation $Px = 0$, we complete the proof.

Note that if c is a constant function, then Theorem 8 is same as Theorem 2.2 of Bae and Sung [6]. Also our Theorem 8 can be compared with Theorem 2.1 of Cramer and Ray [12], and Theorem 3.1 of Altman [3], in which they treat the case $c(t) = t^{-1}B(t)$, where $B : [0,\infty) \to [0,\infty)$ is a continuous nondecreasing function.

By applying Theorem 5, we can obtain a similar result of Theorem 8.

THEOREM 9. *Let D be a nonempty subset of a complete metric space X, Y a Banach space, and let $P : D \to Y$ be a mapping having closed graph in $X \times Y$. Suppose that there are a constant $q \in (0,1)$ and an u.s.c. function $c : [0,\infty) \to [\alpha,\infty)$ with $\alpha > 0$ such that for any $x \in D$, there are an $\bar{x} \in D$ and $\epsilon = \epsilon(x) \in (0,1]$ satisfying*

$$\big\|P\bar{x} - (1-\epsilon)Px\big\| \le q\epsilon\|Px\|$$

and

$$d(x,\bar{x}) \le \epsilon c\big(\|P\bar{x}\|\big)\|Px\|. \tag{19}$$

Then the equation $Px = 0$ has a solution in D. Moreover, if c is nondecreasing, then for any given point $x_0 \in D$, the equation $Px = 0$ has a solution in $D \cap \overline{B}(x_0;r)$, where

$$r = \frac{1}{1-q}\int_0^{\|Px_0\|} c(t)dt.$$

Proof. Let ρ and ϕ be same as in the proof of Theorem 8. Then by the same way of the proof of Theorem 8 and by using the condition (19) instead of (14) we finally obtain that for each $x \in D$, there is an $\overline{x} \in D$ such that

$$\rho(x,\bar{x}) \le \frac{1}{1-q} c\big(\phi(\bar{x})\big)\big\{\phi(x) - \phi(\bar{x})\big\}.$$

Therefore the mapping $g : D \to D$ defined by $gx = \bar{x}$ has a fixed point $z \in D$ by Theorem 3. Moreover, if c is nondecreasing, then by Theorem 5, g has a fixed point $z \in D$ with

$$\begin{aligned} d(x_0, z) \le \rho(x_0, z) &\le \int_0^{\phi(x_0)} \frac{1}{1-q} c(t)dt \\ &= \frac{1}{1-q} \int_0^{\|Px_0\|} c(t)dt. \end{aligned}$$

Since every fixed point of g is a solution of the equation $Px = 0$ by (13), we complete the proof.

We remark that the condition (13) means $-Px \in \Gamma_x(P)$ for any $x \in D$. Therefore the first part of Theorem 8-9 can be proved by using Theorem 6. But our interest about these kinds of mapping theorems is the information concerning the whereabout of the solution.

The assumption that $\inf\{c(t)|t \ge 0\} = \alpha > 0$ is rather restrictive in Theorems 8-9. In fact, if $\alpha = 0$, then we can obtain the following corollary.

COROLLARY 10. *Let D be a nonempty subset of a complete metric space, Y a Banach space, and let $P : D \to Y$ be a mapping having closed graph in $X \times Y$. Suppose that there are a constant $q \in (0,1)$ and an u.s.c. function $c : [0,\infty) \to [0,\infty)$ such that for any $x \in D$, there are an $\overline{x} \in D$ and $0 < \epsilon = \epsilon(x) \le 1$ satisfying the conditions (13) and (14).*

Then the equation $Px = 0$ has a solution in D. Moreover, if c is nonincreasing, then for any given $x_0 \in D$, the equation $Px = 0$ has a solution in $D \cap \overline{B}(x_0; r)$, where r is any real with

$$r > (1-q)^{-1} \int_0^{\|Px_0\|} c(t)dt.$$

Also if P is continuous, then r can be choosen by

$$r = (1-q)^{-1} \int_0^{\|Px_0\|} c(t)dt.$$

Proof. For a fixed $x_0 \in D$, take $\alpha > 0$ such that

$$(1-q)^{-1}\alpha\|Px_0\| < r - (1-q)^{-1}\int_0^{\|Px_0\|} c(t)dt,$$

and put $\bar{c} : [0,\infty) \to [\alpha,\infty)$ with $\bar{c}(t) = c(t) + \alpha$. Then clearly we can apply Theorem 8 for this function $\bar{c}$ instead of c. Therefore, we can obtain the first part of the theorem.

Now suppose that P is continuous. Then the inequality (18) gives the result by applying Theorem 4. This completes the proof.

§4. Ranges of operators

In this section, we obtain the precise estimation of the range of a nonlinear operator by applying our previous theorems. First we give some definitions.

Let X and Y be Banach spaces, and P a mapping from an open subset D of X into Y. We say that P is *Gateaux differentiable* if for each $x \in D$, there is a mapping $dP_x : X \to Y$ satisfying

$$\lim_{t\to 0+} \frac{P(x+th) - P(x)}{t} = dP_x(h), \quad h \in X.$$

If dP_x is a bounded linear operator and if the above limit is attained unformly for all $h \in X$ with $\|h\| = 1$, then P is said to be *Frechet differentiable.* Note that in the definition of Gateaux derivative, we do not require that dP_x is linear, but it follows from the definition that dP_x is homogeneous from the right, that is, $dP_x(th) = tdP_x(h)$ for all $t \geq 0$. Note that Frechet differentiable mappings are necessarily continuous, but Gateaux differentiable mappings, even from R^2 to R need not be continuous.

The nonlinear theory was initiated by Pohozhaev [18] who showed that, if Y is reflexive, $P(X)$ is weakly closed, and if each dP_x is bounded linear with $dP_x(X) = Y$, then $P(X) = Y$. This basic result was considerably sharpened and generalized by Browder [9], Kirk and Caristi [15], Cramer and Ray [12], Ray [19], Bae and Yie [7] and Bae [5] in a series of papers.

THEOREM 11. *Let X and Y be Banach spaces and $P : X \to Y$ a Gateaux differentiable operator from X into Y such that $dP_x(X) = Y$ for each $x \in X$. If $P(X)$ is closed, then P is surjective.*

Proof. Let $x \in X$ and $y \in Y$ be fixed. Since $dP_x(X) = Y$, we have $h \in X$ with $dP_x(h) = y$. This means

$$\lim_{t \to 0+} \frac{P(x+th) - P(x)}{t} = y.$$

Now suppose that $y \neq 0$, and let $q \in (0,1)$ be fixed. Then there is a $t \in (0,1]$ such that

$$\left\| \frac{P(x+th) - P(x)}{t} - y \right\| \leq q\|y\|$$

which says

$$\|P(x+th) - P(x) - ty\| \leq tq\|y\|.$$

Therefore we know that by putting $\overline{x} = x + th, y \in \Gamma_x(P)$. Also if $y = 0$, then by putting $\overline{x} = x, y \in \Gamma_x(P)$. This show that $\Gamma_x(P) = Y$ for each $x \in X$. Therefore by applying Corollary 7, we conclude that P is surjective. This completes the proof.

Note that Theorem 11 includes Theorem 2.1 of Ray [19]. The assumption that P has the closed range is rather strong. Such as the linear theory, this assumption can be weakened by requiring each of the functions dP_x to be an open mapping. In this direction we introduce the following result of Bae and Yie [7].

THEOREM 12 ([7, THEOREM 3.4]). *Let X and Y be Banach spaces and $P : X \to Y$ a Gateaux differentiable mapping having closed graph. Let $c : [0, \infty) \to (0, \infty)$ be a continuous function for which, for each $x \in X$,*

$$\overline{B}(0; c(\|x\|)) \subseteq dP_x(\overline{B}(0;1)).$$

Then for each $K > 0$, $P(B(0;K))$ contains the ball $B(P(0); \int_0^K c(t)dt)$. Moreover, P is an open mapping, and if $\int_0^\infty c(t)dt = \infty$ then P is surjective.

Theorem 12 is a generalization of Theorem 3.1 of Ray and Walker [20] without assuming that c is nonincreasing. The following theorem gives another direction of the above Theorem 12.

THEOREM 13. *Let X and Y be Banach spaces and $P : X \to Y$ a Gateaux differentiable mapping having closed graph. Let $c : [0,\infty) \to (0,\infty)$ be a continuous nondecreasing function for which, for each $x \in X$,*

$$\overline{B}\big(0; c(\|Px\|)^{-1}\big) \subseteq dP_x\big(\overline{B}(0;1)\big).$$

Then for any given $x_0 \in X$ and $r > 0$, $P\big(B(x_0; rc(\|Px_0\| + 2r))\big)$ contains the ball $B\big(P(x_0); r\big)$. In particular, P is an open mapping and $P(X) = Y$.

Proof. It suffices to show that for any $y \in Y$ with $\|P(x_0) - y\| < r$, the equation $Px = y$ has a solution in $B\big(x_0; rc(\|Px_0\| + 2r)\big)$. Let us choose a real $q \in (0,1)$ such that

$$\big\|y - P(x_0)\big\| < (1-q)r.$$

Define a new metric ρ on X such that for $x, y \in X$,

$$\rho(x,y) = \max\big\{\|x-y\|, c(0)(1+q)^{-1}\|Px - Py\|\big\}.$$

Since P has closed graph, (X, ρ) is a complete metric space. Also define $\phi : X \to [0,\infty)$ by $\phi(x) = \|Px - y\|$, so that ϕ is continuous with respect to the metric ρ on X. Now define a mapping $g : X \to X$ as follows ; if $Px = y$, then $gx = x$. Suppose that $Px \neq y$ for $x \in X$. We set

$$v = \|y - Px\|^{-1} c\big(\|Px\|\big)^{-1}(y - Px).$$

Then by (20), there is a $u \in \overline{B}(0;1) \subseteq X$ such that $dP_x(u) = v$ and so, if $h = c(\|Px\|)\|y - Px\|u$, then $dP_x(h) = y - Px$. Since P is Gateaux differentiable, we may choose $t \in (0,1]$ so small that

$$\big\|P(x + th) - P(x) - t dP_x(h)\big\| \le qt\|y - Px\|.$$

By setting $gx = x + th$, this implies

$$\big\|P\big(g(x)\big) - P(x) - t(y - Px)\big\| \le qt\|y - Px\|. \tag{21}$$

and

$$\big\|g(x) - x\| = t\big\|h\| \le tc\big(\|Px\|\big)\|y - Px\|. \tag{22}$$

From (21), we have

$$\|P(g(x)) - P(x)\| \le t(1+q)\|y - Px\|$$

and

$$(1-q)t\|y - Px\| \le \|y - Px\| - \|y - P(g(x))\|,$$

and hence we know that

$$(23)\quad (1+q)^{-1}\|P(g(x)) - P(x)\| \le (1-q)^{-1}\big(\|y - Px\| - \|y - P(gx)\|\big).$$

Also by (22) and the above inequality, we have

$$(24)\qquad \|gx - x\| \le (1-q)^{-1}c\big(\|Px\|\big)\big(\|y - Px\| - \|y - P(gx)\|\big).$$

Therefore from (23) and (24), finally we get

$$\rho(x, gx) \le (1-q)^{-1}c\big(\|Px\|\big)\big\{\phi(x) - \phi(gx)\big\},$$

since $c(0) \le c(\|Px\|)$ for all $x \in X$. Now defined $\overline{c} : [0,\infty) \to [0,\infty)$ by $\overline{c}(t) = c(\|y\| + t)$. Then since $\|Px\| \le \|y\| + \|y - Px\|$, $c(\|Px\|) \le \overline{c}(\phi(x))$ holds. Therefore, we finally obtain

$$\rho(x, gx) \le (1-q)^{-1}\overline{c}\big(\phi(x)\big)\big\{\phi(x) - \phi(gx)\big\}.$$

Since $\overline{c}$ is continuous, by Theorem 3, g has a fixed point $z \in X$ which is actually a solution of the equation $Px = y$ by the inequality (21). Moreover, by using Theorem 1 and the method of the proof of Theorem 3, we can find the fixed point z of g such that there exist finite points $x_1, x_2, \cdots, x_n \in X$ with $x_n = z$ and for each $1 \le i \le n$,

$$\rho(x_{i-1}, x_i) \le (1-q)^{-1}\max\big\{\overline{c}\big(\phi(x_{i-1})\big), \overline{c}\big(\phi(x_i)\big)\big\}\big\{\phi(x_{i-1}) - \phi(x_i)\big\}.$$

But since $\phi(x_i) \le \phi(x_{i-1}) \le \cdots \le \phi(x_0)$ for each i, we have

$$\overline{c}\big(\phi(x_i)\big) \le \overline{c}\big(\phi(x_0)\big) \le c\big(\|y\| + \|y - Px_0\|\big) \le c\big(\|Px_0\| + 2r\big).$$

Hence by using $(1-q)^{-1}\phi(x_0) < r$, we have

$$\begin{aligned} d(x_0, x) \le \rho(x_0, x_n) &\le (1-q)^{-1}c\big(\|Px_0\| + 2r\big)\phi(x_0) \\ &< c\big(\|Px_0\| + 2r\big)r, \end{aligned}$$

which completes the proof.

In fact, in Theorem 13, we need only the continuity of c to obtain the same conclusion.

COROLLARY 14. *Let X and Y be Banach spaces and $P : X \to Y$ a Gateaux differentiable mapping having closed graph. Let $c : [0, \infty) \to (0, \infty)$ be a continuous function satisfying the condition (20). Then P is an open mapping and $P(X) = Y$.*

Proof. Defined $\bar{c} : [0, \infty) \to (0, \infty)$ by $\bar{c}(t) = \sup\{c(r) | 0 \leq r \leq t\}$. Then $\bar{c}$ is a continuous nondecreasing function, and clearly the condition (20) holds for each $x \in X$ if we replace $c(t)$ by $\bar{c}(t)$. Therefore the proof follows from Theorem 13.

The following corollary is a generalization of Theorem 3.4 of Cramer and Ray [12].

COROLLARY 15. *Let X and Y be Banach spaces and $P : X \to Y$ be a Gateaux differentiable mapping having closed graph. Suppose that for each $t > 0$, there is a $c(t) > 0$ such that whenever $\|x\| \leq t$,*

$$\overline{B}\big(0; c(t)^{-1}\big) \subseteq dP_x\big(\overline{B}(0; 1)\big),$$

and that $\|Px\| \to \infty$ as $\|x\| \to \infty$. Then P is an open mapping and $P(X) = Y$.

Proof. Defined $c_1 : [0, \infty) \to (0, \infty)$ by putting

$$c_1(t) = \sup\big\{r > 0 | \overline{B}(0; r^{-1}) \subseteq dP_x\big(\overline{B}(0; 1)\big), \quad \text{for } \|Px\| \leq t\big\}.$$

Then since $\|Px\| \to \infty$ as $\|x\| \to \infty$, by assumption we know that $c_1(t) > 0$ for all $t \geq 0$ and c_1 is a nondecreasing function. Now we can choose a continuous nondecreasing function $c_2 : [0, \infty) \to (0, \infty)$ such that $c_1(t) \leq c_2(t)$ for all $t \geq 0$. Then for this function c_2 instead of c_1, it can be easily shown that the condition (20) is satisfied for all $x \in X$. Therefore the proof follows from Theorem 13. This completes the proof.

References

1. M. Altman, *Contractor direction, directional contractors and directional contractions for solving equations*, Pacific J.Math. **62** (1976), 1–18.
2. M. Altman, *Contractors and contractor directions, theory and applications*, Marcel Dekker, New York and Basel (1977).
3. M. Altman, *A generalization of the Brezis-Browder principle on ordered sets*, Nonlinear Analysis, TMA **6** (1982), 157–165.

4. M. Altman, *Weak contractor directions and weak directional contractors*, Nonlinear Analysis, TMA **7** (1983), 1043–1049.
5. J.S. Bae, *Mapping theorems for nonlinear operators*, J. Korean Math. **25** (1988), 289–301.
6. J.S. Bae and N.S. Sung, *Ranges of sums of two nonlinear operators*, Math. Japonica **6** (1989), 851–864.
7. J.S. Bae and S. Yie, *Ranges of Gateaux differentiable operators and local expansions*, Pacific J.Math. **125** (1986), 289–300.
8. H. Brezis and F.E. Browder, *A general principle on ordered sets in nonlinear functional analysis*, Advance in Math. (1976), 355-364.
9. F.E. Browder, *Normal solvability for nonlinear mappings in Banach spaces*, Bull. Amer. Math. Soc. **77** (1971), 73–77.
10. F.E. Browder, *Nonlinear operators and nonlinear equation of evolution in Banach spaces*, Proc. Symp. Pure Math. vol.18, Pt.2, AMS, Providence R. (1976).
11. J. Caristi, *Fixed point theorems for mappings satisfying inwardness conditions*, Trans. Amer. Math. Soc. **215** (1976), 241–251.
12. W.J. Cramer and W.O. Ray, *Solvability of nonlinear operator equations*, Pacific J.Math. **95** (1981), 37–50.
13. I. Ekeland, *Sur les problems variationnels*, Compte Rendus Acad. Soc. Paris **275** (1972), 1057–1059.
14. I. Ekeland, *Nonconvex minimizaton problems*, Bull. Amer. Soc. Math. **1** (1976), 443-474.
15. W.A. Kirk and J. Caristi, *A mapping theorem in metric and Banach spaces*, Bull. Acad. Pol. Sci. Math. Astron. et Phys. **23** (1975), 891–894.
16. J.A. Park and J.S. Bae, *A surjectivity theorem for a nonlinear operator with Altman's directional contractions*, Bull. Korean Math. Soc. **24** (1987), 131–134.
17. S. Park and J.S. Bae, *On the Ray-Walker extension of the Caristi-Kirk fixed point theorem*, Nonlinear Analysis, TMA **9** (1985), 1135-1136.
18. S.I. Pohozhayev, *On the normal solvability of nonlinear operators*, Dokl. Akad. Nauk USSR **184** (1969), 40–43.
19. W.O. Ray, *Normally solvable nonlinear operators*, Contemp. Math. **18** (1983), 155–165.
20. W.O. Ray and A. Walker, *Mapping theorems for accretive and differentiable operators*, Nonlinear Analysis, TMA **6** (1982), 423–433.
21. I. Rosenholtz and W.O. Ray, *Mapping theorems for differentiable operators*, Bull. Acad. Polon. Sci. Ser. Math. Astron. et Phys. **29** (1981), 265–273.

Myong-Ji University
Yongin 449-800, Korea

Seoul Poly Technical University
Seoul 139-743, Korea