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MULTIPLE SOLUTIONS OF A CLASS OF SINGULAR PERTURBATION PROBLEMS WITH NEUMANN TYPE

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1. Introduction

In this paper we discuss the existence of multiple nontrivial solutions and interior transition layers of those solutions of a class of semilinear elliptic singular perturbation problems with Neumann type:

$$\begin{aligned} \epsilon^2 \Delta u + f(x,u) &= 0, \qquad x \in \Omega, \\ \frac{\partial u}{\partial n} &= 0, \qquad x \in \partial\Omega, \end{aligned} \tag{I}$$

where $\frac{\partial u}{\partial n}$ denotes the outward normal derivative of u on $\partial\Omega$, and we assume that $\Omega \subset R^n$ (n≥ 1) is a smooth bounded open set, $\partial\Omega \in C^{2,\alpha}$ $(0<\alpha<1)$ and $\bar{\Omega} = \Omega \cup \partial\Omega$.

In section 2, we assume that $f : \bar{\Omega} \times R \to R$ satisfies the following:

(1) $f \in C^1(\bar{\Omega} \times R)$

(2) There exist exactly three functions $g_i : \bar{\Omega} \to R$ $(i = 0,1,2)$ which belong to $C^2(\bar{\Omega})$, and

$$g_1(x) < g_0(x) < g_2(x)$$

and

$$f(x, g_i(x)) = 0$$

for all $x \in \bar{\Omega}$.

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(3) There exists a positive constant K such that

$$f_u(x, g_i(x)) < -K$$

for all $x \in \bar{\Omega}$ and $i = 1, 2$.

(4) There exist two nonempty disjoint connected open subsets Ω_1 and Ω_2 of Ω such that for any point $x \in \partial\Omega_1$,

$$\int_{\gamma(x)}^{g_2(x)} f(x,u)\,du > 0$$

for all $\gamma(x) \in \big[g_1(x), g_2(x)\big)$, and for any $x \in \partial\Omega_2$,

$$\int_{g_1(x)}^{\gamma(x)} f(x,u)\,du < 0$$

for all $\gamma(x) \in (g_1(x), g_2(x)]$. We also assume that $\partial\Omega_1$ and $\partial\Omega_2$ are smooth.

Constructing three pairs of quasisupersolutions and quasisubsolutions of the problem (I), we prove that there is ϵ_0 such that for any ϵ with $0 < \epsilon \leq \epsilon_0$ (I) has five distinct ordered solutions in $C^2(\bar{\Omega})$ between g_1 and g_2 and one of them has an interior transition layer as $\epsilon \to 0$.

In [1], the limit of global minimizers of functionals of the type

$$\frac{\epsilon^2}{2}\int_\Omega |\nabla u|^2\,dx + \int_\Omega F(|x|, u)\,dx\,,$$

as $\epsilon \to 0$, where $F(|x|, z) = \int_{r(|x|)} z f(|x|, u)\,du$, has been discussed previously in case the domain is an interval in R^1, or a ball, or an annulus in R^n by the variational method.

We also discuss interior transition layers of solutions in the case of $f(x,u) = u(u - a(x))(1-u)$ and we show that several types of interior transition layers of solutions are dependent on the shape of the graph of the function $a(x)$.

In the section 3, we discuss the existence of multiple solutions of (I) in the case of $f(x,u) = g(x)h(u)$ for sufficiently small $\epsilon > 0$. In the problem we assume that h has at least three zeros $z_1 < z_2 < z_3$ and $h'(z_1) < 0$, $h'(z_2) > 0$, and $h'(z_3) < 0$, and the function g takes both positive values and negative values on the domain Ω. Constructing two pairs of quasisupersolutions and quasisubsolutions of (I), we show that there is ϵ_0 such that for any ϵ with $0 < \epsilon \leq \epsilon_0$ the problem (I) has at least two distinct nontrivial solutions which lie between z_1 and z_3 and which have interior transition layers as $\epsilon \to 0$. We also prove that the existence of a nontrivial solution having the interior transition layer even though h has only two zeros.

In [2], in the case of $f(x,u) = g(x)h(u) = g(x)u(1-u)\big[k(1-u)+(1-k)u\big]$, $0 < k < 1$, and with condition $\int_\Omega g(x)\,dx < 0$ or $\int_\Omega g(x)\,dx > 0$, they proved by the bifurcation method that the existence of a nontrivial positive solution of (I) between 0 and 1.

Without the integral conditions for the function g, we prove that for all sufficiently small $\epsilon > 0$ the problem (I) has two nontrivial positive solutions and interior transition layers for the solutions according to the value of k if $k \neq 0$, $k \neq \frac{1}{2}$, and $k \neq 1$. We also prove that if $k = 0$, or $k = \frac{1}{2}$, or $k = 1$, then the problem (I) has at least one nontrivial positive solution having an interior transition layer as $\epsilon \to 0$.

2. Main Results

DEFINITIONS. A function $w : \bar{\Omega} \to R$ is a quasisupersolution (or quasisubsolution) of (I) if for any $x_0 \in \bar{\Omega}$, there exist a neighborhood N of x_0 and a finite number of functions $w_k \in C^2(N)$, $k = 1, 2, \cdots, p$ such that

$$w(x) = \min_{1 \leq k \leq p} w_k(x) \quad \big(\text{or} \quad \max_{1 \leq k \leq p} w_k(x)\big)$$

for all $x \in N$, where p may depend on x_0, and

$$\epsilon^2 \Delta w_k(x) + f\big(x, w_k(x)\big) \leq 0 \quad (\text{or} \quad \geq 0)$$

for all $x \in N \cap \Omega$ and $k = 1, 2, \cdots, p$. Furthermore, if $x_0 \in \partial\Omega$,

$$\frac{\partial w_k}{\partial n}(x) \geq 0 \quad (\text{or} \quad \leq 0)$$

for all k.

Constructing three pairs of quasisubsolutions and quasisupersolutions of (I) and using Theorem 3 in [4], we prove the existence of five distinct ordered solutions of (I) and interior transition layers of those solutions as $\epsilon \to 0$.

Theorem 3 in [4] is as follows:

THEOREM 2.1 [4]. *Suppose that* $\bar{w}_1, \bar{w}_2$ *are quasisubsolutions and* $\hat{w}_1, \hat{w}_2$ *are quasisupersolutions of elliptic problems of the type:*

$$\Delta u(x) + f\big(x, u(x)\big) = 0, x \in \Omega,$$

$$\text{(II)} \qquad p(x)u(x) + q(x)\frac{\partial u}{\partial n} = \varphi(x), \qquad x \in \partial\Omega$$

such that $\bar{w}_1(x) \le \hat{w}_2(x)$ *and* $\bar{w}_j(x) \le \hat{w}_j(x)$ *for all* $x \in \bar{\Omega}$ *and* $j = 1, 2$, *and* $\bar{w}_2(x_0) > \hat{w}_1(x_0)$ *for some* $x_0 \in \bar{\Omega}$. *If* $\hat{w}_1$ *and* $\bar{w}_2$ *are not solutions of* (II), *then* (II) *has at least three distinct solutions* u_j *in* $C^2(\bar{\Omega})$, $(j = 0, 1, 2)$, *such that*

$$\bar{w}_j(x) \le u_j(x) \le \hat{w}_j(x), \quad u_1(x) \le u_0(x) \le u_2(x)$$

for all $x \in \bar{\Omega}$ *and* $j = 1, 2$, *and especially*

$$u_0 \in [\bar{w}_1, \hat{w}_2] \setminus [\bar{w}_1, \hat{w}_1] \cup [\bar{w}_2, \hat{w}_2].$$

REMARK. In Theorem 2-1, the notation $[\bar{w}_i, \hat{w}_j] = \big\{u \in C(\bar{\Omega}) : \bar{w}_i(x) \le u(x) \le \hat{w}_j(x), x \in \bar{\Omega}\big\}$, $i \le j$.

The function f satisfies that $f \in C^\alpha(\bar{\Omega} \times I)$ (Here $0 < \alpha < 1$, I is a fixed finite closed interval in R) and there is a positive number M such that $|f(x,u) - f(x,v)| \le M|u - v|$ for all $x \in \bar{\Omega}$ and $u, v \in I$.

φ and $\partial\Omega$ are smooth.

In order to construct a quasisubsolution and a quasisupersolution of (I), we use a coordinate transformation near the boundary $\partial\Omega_1$ and $\partial\Omega_2$. If $x \in \Omega_i$, $(i = 1, 2)$ we denote by $t = t(x)$ the distance from x to $\partial\Omega_i$ and by $s = s(x)$ the point of $\partial\Omega_i$ which is closest to x. Of course $s(x)$ might not be uniquely defined, but will be if x is close enough to $\partial\Omega_i$. Let O_i be the open set of all points of Ω_i such that the normal through distinct points of $\partial\Omega_i$ do not intersect on O_i. We denote $x = (s,t)$ if $x \in O_i$ and $x = s = (s,0)$ if $x \in \partial\Omega_i$, and if $x \in O_i$, we have

$$\epsilon^2 \Delta_x = \epsilon^2 \frac{\partial^2}{\partial t^2} + 0(\epsilon).$$

by the substitution $x = (s,t)$.

Now consider the following boundary value problem:

$$\begin{cases} \frac{\partial^2 u}{\partial \tau^2} & = F(s, u(s,\tau)) \\ u(s,0) & = \xi(s), \quad u(s,\infty) = 0\,, \end{cases}$$

where $s \in \partial\Omega$ is a parameter and $\tau \in [0,\infty)$ and $F(s,u)$ is a real valued function.

The following fact is well known.

LEMMA [3]. *Let $\xi(s)$ and $F(s,u)$ be infinitely differentiable for all $s \in \partial\Omega$ and $u \in (-\infty,+\infty)$, all derivatives being uniformly continuous. For all $s \in \partial\Omega$, assume that*

$$F(s,0) = 0, \quad F_u(s,0) > 0, \quad \int_0^w F(s,u)\,du > 0$$

for all $w \in (0,\xi(s)]$ or $[\xi(s),0)$. Then there is a unique monotone solution $v(s,\tau)$ of the above boundary value problem and it is infinitely differentiable in s and τ. Moreover each of the derivatives of v decays exponentially as $\tau \to \infty$, uniformly in s, in the sense that if D is any C^∞ linear differential operator in s and τ, then there exist positive constants C and β, possibly depending on D, such that $|Dv(s,\tau)| \le ce^{-\beta\tau}$.

Hence, we can say that

$$\epsilon^2 \Delta_x v(s,\tau) = \frac{\partial^2 v}{\partial \tau^2} + 0(\epsilon)$$

as $\epsilon \to 0$, uniformly on $s \in \partial\Omega$, if $v(s,\tau)$ is the unique monotone solution of the above boundary value problem.

Using Lemma, we have the following theorem which yields the existence of a pair of quasisubsolution and quasisupersolution of (I).

THEOREM 2-2. *Assume (1)-(4). Then there is $\epsilon_0 > 0$ such that for any ϵ with $0 < \epsilon \le \epsilon_0$ there exist a quasisub solution $\bar{w}(x;\epsilon)$ and a quasisuper solution $\hat{w}(x;\epsilon)$ of (I) so that $\bar{w}(x;\epsilon) \le \hat{w}(x;\epsilon)$ for all $x \in \bar{\Omega}$,*

$$\lim_{\epsilon \to 0} \bar{w}(x;\epsilon)\big(\text{or}\,\hat{w}(x;\epsilon)\big) = \begin{cases} g_2(x) & \text{in} \quad \Omega_1 \\ g_1(x) & \text{in} \quad \Omega_2\,, \end{cases}$$

and the convergence is uniform on every compact subset of indicated regions.

Proof. We first construct a quasisupersolution $\hat{w}(x;\epsilon)$ of (I). From the hypothesis (3), there is a positive constant r_1 such that

$$f_u(x, g_i(x) \pm r) \le -K < 0$$

for all $0 \le r \le r_1$ and for all $x \in \bar{\Omega}$, $i = 1, 2$. Let $\rho_0 > 0$ be a sufficiently small number so that $\rho_0 < r_1$, and let $L(s, \rho_0) = f(s, g_1(s) + \rho_0)$ for all $s \in \partial\Omega_2$. Then $L(s, \rho_0) < 0$ for all $s \in \partial\Omega_2$. Let $F(s, u) = -f(s, g_1(s) + \rho_0 + u) + L(s, \rho_0)$. Then $F(s, 0) = 0$ and $F_u(s, 0) \ge K > 0$ for all $s \in \partial\Omega_2$. Now, since

$$\begin{aligned} &\int_0^{\beta(s)} F(s, u)\, du \\ &= \int_{g_1(s)+\rho_0}^{g_1(s)+\rho_0+\beta(s)} -f(s, u)\, du + L(s, \rho_0)\beta(s). \end{aligned}$$

and since if we choose ρ_0 so that

$$\int_{g_1(s)+\rho_0}^{g_2(s)+\rho_0} -f(s, u)\, du > 0$$

for all $s \in \partial\Omega_2$, then

$$\int_{g_1(s)+\rho_0}^{g_1(s)+\rho_0+\beta(s)} -f(s, u)\, du > 0$$

for all $\beta(s) \in (0, g_2(s) - g_1(s)]$. Since

$$\lim_{\rho_0 \to 0} L(s, \rho_0) = 0,$$

uniformly for all $s \in \partial\Omega_2$,

$$\int_0^{\beta(s)} F(s,u)\,du > 0$$

for all $\beta(s) \in (0, g_2(s) - g_1(s)]$. Then, by Lemma, there is a unique monotone solution $v(s,\tau)$ of the problem

$$\begin{aligned}\frac{\partial^2 v}{\partial \tau^2} &= -f\big(s, g_1(s) + \rho_0 + v\big) + L(s,\rho_0)\\ v(s,0) &= g_2(s) - g_1(s), \quad v(s,\infty) = 0\end{aligned}$$

such that v and the first derivatives and second derivatives of v in s and τ decay exponentially as $\tau \to \infty$. Let $t = \epsilon\tau$ and $V(s,t) = v(s, \frac{t}{\epsilon})$, and let $\bar{O}_\kappa = \big\{(s,t) \in \bar{\Omega}_2 : 0 \le t \le \kappa,\ s \in \partial\Omega_2\big\}$, where κ is so chosen that the normals through distinct points of $\partial\Omega_2$ do not intersect on $\bar{O}_\kappa$. Then if $x = (s,t) \in O_\kappa$,

$$\begin{aligned}\epsilon^2 \Delta V(x) &= \epsilon^2 \frac{\partial^2 V}{\partial t^2} + 0(\epsilon)\\ &= \frac{\partial^2 v}{\partial \tau^2} + 0(\epsilon)\\ &= -f\big(s, g_1(s) + \rho_0 + v(s,\tau)\big) + L(s,\rho_0) + 0(\epsilon)\\ V(s,0) &= v(s,0), \quad V(s,\infty) = 0\end{aligned}$$

We take a smooth function $\sigma(t) \in C^\infty([0,\infty))$ such that $\sigma(t) = 1$ for $0 \le t \le \frac{\kappa}{3}$ and $\sigma(t) = 0$ for $\frac{2}{3}\kappa \le t$, and $0 \le \sigma(t) \le 1$ for all $t \ge 0$, and we define $\hat{V}(s,t) = V(s,t)\sigma(t)$ for $(s,t) \in \bar{O}_\kappa$ and $\hat{V}(x) = 0$ for $x \in \Omega_2 \setminus \bar{O}_\kappa$. Then the function $\hat{V}$ is in $C^2(\bar{\Omega}_2)$. We note that κ is independent of ϵ.

Let $\hat{U}(x;\epsilon) = g_1(x) + \rho_0 + \hat{V}(x)$ for all $x \in \bar{\Omega}_2$. Now we prove that

$$\epsilon^2 \Delta \hat{U}(x;\epsilon) + f(x, \hat{U}(x;\epsilon)) \le 0$$

in Ω_2. For any $x=(s,t)$ with $0<t\le \frac{\kappa}{3}$,

$$
\begin{aligned}
&\epsilon^2\Delta\hat{U}(x;\epsilon)+f\big(x,\hat{U}(x;\epsilon)\big)\\
=&\epsilon^2\Delta\hat{V}(x)+f\big(x,\hat{U}(x;\epsilon)\big)+0(\epsilon^2)\\
=&\frac{\partial^2 v}{\partial\tau^2}+f\big(x,\hat{U}(x;\epsilon)\big)+0(\epsilon)\\
=&-f\big(s,g_1(s)+\rho_0+v(s,\tau)\big)+f\big(x,g_1(s)+\rho_0+v(s,\tau)\big)\\
&-f\big(x,g_1(s)+\rho_0+v(s,\tau)\big)+f\big(x,g_1(x)+\rho_0+v(s,\tau)\big)\\
&+L(s,\rho_0)+0(\epsilon)\\
=&\nabla_x f\big(x^*,g_1(s)+\rho_0+v(s,\tau)\big)\circ\Big[\frac{dx}{dt}(t^*)\Big]t\\
&+f_u(x,u^*)\Big(\nabla g_1(x^{**})\circ\frac{dx}{dt}(t^{**})\Big)t+L(s,\rho_0)+0(\epsilon),
\end{aligned}
$$

where x^* and x^{**} are on the line segment passing through the point x and s on O_κ, and u^* is between $\hat{U}(x;\epsilon)$ and $g_1(s)+\rho_0+v(s,\tau)$, and $0<t^*,t^{**}<t$, and $\circ$ is the inner product in R^n. Hence

$$
\epsilon^2\Delta\hat{U}(x;\epsilon)+f\big(x,\hat{U}(x;\epsilon)\big)=0(t)+0(\epsilon)+L(s,\rho_0).
$$

If ϵ and κ are sufficiently small, then

$$
\epsilon^2\Delta\hat{U}(x;\epsilon)+f\big(x,\hat{U}(x;\epsilon)\big)\le 0.
$$

For any $x=(s,t)$ with $\frac{\kappa}{3}<t<\frac{2}{3}\kappa$,

$$
\begin{aligned}
&\epsilon^2\Delta\hat{U}(x;\epsilon)+f\big(x,\hat{U}(x;\epsilon)\big)\\
=&\epsilon^2\Delta\hat{V}(s,t)+f\big(x,g_1(x)+\rho_0+\hat{V}(s,t)\big)+0(\epsilon^2)\\
=&\frac{\partial^2 v}{\partial\tau^2}\sigma+f\big(x,g_1(x)+\rho_0+V\sigma\big)+0(\epsilon).
\end{aligned}
$$

Since $\dfrac{\partial^2 v}{\partial\tau^2}$ and v decay exponentially as $\tau\to\infty$ and $f\big(x,g_1(x)+\rho_0\big)<0$, so

$$
\epsilon^2\Delta\hat{U}(x;\epsilon)+f(x,\hat{U}(x;\epsilon))\le 0
$$

if ϵ is sufficiently small. For any $x \in \Omega_2 \setminus \bar{O}_{\frac{3}{8}\kappa}$,

$$\hat{U}(x;\epsilon) = g_1(x) + \rho_0.$$

Then

$$\begin{aligned}&\epsilon^2 \Delta \hat{U}(x;\epsilon) + f\big(x, \hat{U}(x;\epsilon)\big)\\ =&0(\epsilon^2) + f\big(x, g_1(x) + \rho_0\big) \le 0\end{aligned}$$

if ϵ is sufficiently small. Furthermore, we note that if $x \in \partial\Omega_2$,

$$\hat{U}(x;\epsilon) = g_2(x) + \rho_0.$$

We choose a function $h \in C^2(\bar{\Omega})$ such that $\dfrac{\partial h}{\partial n} \ge 0$ on $\partial\Omega$ and $g_2(x) < h(x) \le g_2(x) + \rho_0$ in $\bar{\Omega}$. Let

$$\hat{w}(x;\epsilon) = \begin{cases} \min\{\hat{U}(x;\epsilon), h(x)\} & \text{if} \quad x \in \bar{\Omega}_2 \\ h(x) & \text{if} \quad x \in \bar{\Omega} \setminus \bar{\Omega}_2\,. \end{cases}$$

Since $f(x, h(x)) < 0$ for all $x \in \bar{\Omega}$,

$$\epsilon^2 \Delta \hat{w}(x;\epsilon) + f(x, \hat{w}) \le 0$$

if $\hat{w}$ is twice partial differentiable at x in Ω. Moreover, if $x = (s, 0) \in \partial\Omega$, then

$$\frac{\partial \hat{w}}{\partial n}(x) = \frac{\partial h}{\partial n}(x) \ge 0\,.$$

We hence note that $\hat{w}(x;\epsilon)$ is a quasisupersolution of (I) if $\epsilon > 0$ is sufficiently small.

To construct a quasisubsolution of (I), we let $G(s,u) = f(s, g_2(s) - \nu_0 - u) - l(s, \nu_0)$, where $l(s, \nu_0) = f(s, g_2(s) - \nu_0)$ for all $s \in \partial\Omega_1$ and for some ν_0 with $0 < \nu_0 < r_1$. Then $l(s, \nu_0) > 0$, $G(s, 0) = 0$, $G_u(s, 0) \ge K$, and

$$\int_0^{\alpha(s)} G(s,u)\,du > 0$$

for all $\alpha(s) \in (0, g_2(s) - g_1(s)]$ if ν_0 is sufficiently small.

By the similar method used in constructing quasisupersolution of (I), we can find a quasisubsolution $\bar{w}(x;\epsilon)$ of (I) having the following properties:

$$\bar{w}(x;\epsilon) = \begin{cases} \max\{\bar{U}(x;\epsilon), k(x)\} & textif \quad x \in \bar{\Omega}_1 \\ k(x) & \text{if} \quad x \in \bar{\Omega} \setminus \bar{\Omega}_1, \end{cases}$$

$$\bar{w}(x;\epsilon) = g_1(s) - \nu_0 \quad \text{for all} \quad x \in \partial\Omega_1 \,,$$

$$k \in C^2(\bar{\Omega}),\ g_1(x) - \nu_0 \le k(x) < g_1(x) \text{ in } \bar{\Omega},\ \frac{\partial k}{\partial n} \le 0 \text{ on } \partial\Omega \,,$$

$$\bar{U}(x;\epsilon) = g_2(x) - \nu_0 + \bar{v}(x) \quad \text{for all} \quad x \in \bar{\Omega}_1 \,,$$

$$\bar{v}(x;\epsilon) = \begin{cases} U(s,t)\sigma(t) & \text{for} \quad (s,t) \in \bar{O}_\eta \\ 0 & \text{for all} \quad x \in \Omega_1 \setminus \bar{O}_\eta \,, \end{cases}$$

$$\bar{O}_\eta = \left\{(s,t) \in \bar{\Omega}_1 : 0 \le t \le \eta,\ s \in \partial\Omega_1\right\} \,.$$

Here η is so chosen that the normals through distinct points of $\partial\Omega_1$ do not intersect on $\bar{O}_\eta$ and η is independent of ϵ. As before $\sigma(t) \in C^\infty([0,\infty))$, $\sigma(t) = 1$ for $0 \le t \le \frac{\eta}{3}$ and zero for $\frac{2}{3}\eta \le t$, and $0 \le \sigma(t) \le 1$ for all $t \ge 0$.

Finally, $U(s,t) = u(s,\tau)$, $(t = \epsilon\tau)$, is the unique monotone solution of the problem:

$$\frac{\partial^2 u}{\partial \tau^2} = f\big(s, g_2(s) - \nu_0 - u\big) - l(s,\nu_0)$$

$$u(s,0) = g_2(s) - g_1(s), \quad u(s,\infty) = 0$$

Clearly, $\bar{w}(x;\epsilon) \le \hat{w}(x;\epsilon)$ for all $x \in \bar{\Omega}$ and the convergence for $\bar{w}$ and $\hat{w}$ as $\epsilon \to 0$ is true if we choose ρ_0 and ν_0 smaller and smaller. This completes the proof.

Choosing four functions $p_i(x)$, $(i = 1,2,3,4)$ such that $p_i \in C^2(\bar{\Omega})$, for each $x \in \partial\Omega$,

$$\frac{\partial p_1}{\partial n} \le 0, \quad \frac{\partial p_2}{\partial n} \ge 0, \quad \frac{\partial p_3}{\partial n} \le 0, \quad \frac{\partial p_4}{\partial n} \ge 0,$$

and for each $x \in \bar{\Omega}$,

$$f\big(x,p_1(x)\big) > 0,\ f\big(x,p_2(x)\big) < 0,\ f\big(x,p_3(x)\big) > 0,\ f\big(x,p_4(x)\big) < 0 \,,$$

$$g_1(x) - \nu_0 \le p_1(x) \le g_1(x) \le p_2(x) \le g_1(x) + \nu_0 \,,$$

$$g_2(x) - \rho_0 \le p_3(x) \le g_2(x) \le p_4(x) \le g_2(x) + \rho_0 \,,$$

and using Theorem 2-1 and Theorem 2-2, we conclude the following theorem which is the main result.

THEOREM 2-3. *With assumptions (1)-(4), there is* $\epsilon_0 > 0$ *such that for any* ϵ *with* $0 < \epsilon \leq \epsilon_0$ *the problem* (I) *has at least five distinct ordered solutions* $u_j(x;\epsilon)$ *in* $C^2(\bar{\Omega})$ *such that*

$$
\begin{aligned}
p_1(x) &\leq u_1(x;\epsilon) \leq p_2(x)\,, \\
u_1(x;\epsilon) &\leq u_2(x;\epsilon) \leq u_3(x;\epsilon)\,, \\
\bar{w}(x;\epsilon) &\leq u_3(x;\epsilon) \leq \hat{w}(x;\epsilon)\,, \\
p_3(x) &\leq u_5(x;\epsilon) \leq p_4(x)\,, \\
u_3(x;\epsilon) &\leq u_4(x;\epsilon) \leq u_5(x;\epsilon)\,,
\end{aligned}
$$

for all $x \in \bar{\Omega}$. *Especially*

$$
\begin{aligned}
&\lim_{\epsilon \to 0} u_j(x;\epsilon) = g_j(x) \quad x \in \bar{\Omega} \quad (j = 1,5)\,, \\
&\lim_{\epsilon \to 0} u_2(x;\epsilon) = g_1(x) \quad x \in \Omega_2\,, \\
&\lim_{\epsilon \to 0} u_4(x;\epsilon) = g_2(x) \quad x \in \Omega_1\,, \\
&\lim_{\epsilon \to 0} u_3(x;\epsilon) = \begin{cases} g_1(x) & x \in \Omega_2 \\ g_2(x) & x \in \Omega_1\,, \end{cases}
\end{aligned}
$$

and the above convergencies are uniform on every compact subset of indicated regions.

REMARK. In fact, the above theorem is true in case that $f \in C^{\alpha}(\bar{\Omega} \times I)$, $0 < \alpha < 1$, and I is a bounded interval in R such that $g_i(x) \in I$ for all $i = 1,3$ and for all $x \in \bar{\Omega}$, if we replace the assumption (3) by the hypothesis that there is a positive number r_1 such that

$$
f\big(x, g_i(x) - r\big) > 0 \quad \text{and} \quad f\big(x, g_i(x) + r\big) < 0
$$

for all $x \in \bar{\Omega}$, for all r with $0 < r \leq r_1$, and $i = 1,3$.

From Theorem 2-3 and Remark, we have the following results:

COROLLARY 1. *In the problem (I), let* $f(x,u) = u(u-a(x))(1-u)$. *If the function* $a(x)$ *has the following properties:* $0 < a(x) < 1$ *for all* $x \in \bar{\Omega}$, $a \in C^1(\bar{\Omega})$, *and there are two nonempty disjoint connected open subsets* Ω_1 *and* Ω_2 *with smooth boundaries such that for any point* $x \in \partial\Omega_1$,

$$\int_{\gamma(x)}^{1} u\big(u - a(x)\big)(1-u)\, du > 0$$

for all $\gamma(x) \in [0,1)$, *and for any* $x \in \partial\Omega_2$

$$\int_{0}^{\gamma(x)} u\big(u - a(x)\big)(1-u)\, du < 0$$

for all $\gamma(x) \in (0,1]$, *then for all sufficiently small* $\epsilon > 0$ *the problem (I) has at least three distinct positive nontrivial ordered solutions* $u_j(x;\epsilon)$ *in* $C^2(\bar{\Omega})$ *such that*

$$0 < u_1(x;\epsilon) \le u_2(x;\epsilon) \le u_3(x;\epsilon) < 1$$

for all $x \in \bar{\Omega}$,

$$\begin{aligned}
&\lim_{\epsilon \to 0} u_1(x;\epsilon) = 0 \qquad x \in \Omega_2\,,\\
&\lim_{\epsilon \to 0} u_3(x;\epsilon) = 1 \qquad x \in \Omega_1\,,\\
&\lim_{\epsilon \to 0} u_2(x;\epsilon) = \begin{cases} 1 & x \in \Omega_1\,,\\ 0 & x \in \Omega_2\,, \end{cases}
\end{aligned}$$

and the above convergencies are uniform on every compact subset of indicated regions.

COROLLARY 2. *Let* $f(x,u) = u(u-a(x))(1-u)$, $0 < a(x) < 1$ *for all* $x \in \bar{\Omega}$, *and* $a \in C^1(\bar{\Omega})$. *If there are* N *nonempty disjoint connected open subsets* $\Omega_1, \Omega_2, \cdots, \Omega_N$ *of* Ω *such that their boundaries are smooth, for any point* $x \in \partial\Omega_i$ *(i's are odd)*

$$\int_{\gamma(x)}^{1} f(x,u)\, du > 0$$

for all $\gamma(x) \in [0,1)$, *and for any point* $x \in \partial\Omega_j$ *(j's are even)*

$$\int_0^{\gamma(x)} f(x,u)\,du < 0$$

for all $\gamma(x) \in (0,1]$, *then for all sufficiently small* $\epsilon > 0$ *the problem (I) has three distinct nontrivial positive ordered solutions* $u_k(x;\epsilon)$ *in* $C^2(\bar{\Omega})$ *such that*

$$0 < u_1(x;\epsilon) \le u_2(x;\epsilon) \le u_3(x;\epsilon) < 1$$

for all $x \in \bar{\Omega}$,

$$\lim_{\epsilon \to 0} u_1(x;\epsilon) = 0 \quad x \in \Omega_j \quad (j = \text{even}),$$
$$\lim_{\epsilon \to 0} u_3(x;\epsilon) = 1 \quad x \in \Omega_i \quad (i = \text{odd}),$$
$$\lim_{\epsilon \to 0} u_2(x;\epsilon) = \begin{cases} 1 & x \in \Omega_i \quad (i = \text{odd}) \\ 0 & x \in \Omega_j \quad (j = \text{even}), \end{cases}$$

and the above convergencies are uniform on every compact subset of indicated regions.

3. The function $f(x,u) = g(x)h(u)$ case

Consider a model with two alleles A_1 and A_2 corresponding to three possible genotypes A_1A_1, A_1A_2, A_2A_2. The population lives in a region Ω in R^n. Let $u(x,t)$ denote the frequency of the allele A_1 at time t at the point x in Ω. Changes in gene frequency are assumed to be caused only by the flow of genes within Ω and selective advantages for certain genotypes in certain subregions of Ω.

In [2], they said that u satisfies the semilinear parabolic equation

$$u_t(x,t) = \epsilon^2 \Delta u + g(x)h(u)$$

in Ω, where $h(u) = u(1-u)[k(1-u) + (1-k)u]$, for some constants $\epsilon > 0$, and $0 < k < 1$. They assumed that g takes on both positive and negative values on the region Ω and that either

$$\int_\Omega g(x)\,dx < 0$$

or

$$\int_{\Omega} g(x)\,dx > 0\,.$$

They then proved by bifurcation method that there is $\epsilon_0 > 0$ such that for all ϵ with $0 < \epsilon \leq \epsilon_0$ the problem (I) has a nontrivial positive solution $u(x;\epsilon)$ in $C^2(\bar{\Omega})$ such that $0 < u(x;\epsilon) < 1$ for all $x \in \bar{\Omega}$.

Now, without the integral condition for g we prove the existence of two nontrivial positive solutions as well as the existence of interior transition layers of the solutions as $\epsilon \to 0$. Here we assume that $g \in C^1(\bar{\Omega})$ and $h \in C^1(I)$, where I is an interval in R.

THEOREM 3.1. *Let $f(x,u) = g(x)h(u)$, h have three zeros $z_1 < z_2 < z_3$ and no other zeros between z_1 and z_3, and h satisfy that $h'(z_1) < 0$, $h'(z_2) > 0$, and $h'(z_3) < 0$. If $\Omega_1 = \{x \in \Omega : g(x) < 0\}$ and $\Omega_2 = \{x \in \Omega : g(x) > 0\}$ are connected open subsets of Ω with smooth boundaries, then there is ϵ_0 such that for all ϵ with $0 < \epsilon \leq \epsilon_0$ the problem (I) has two nontrivial ordered solutions $u_i(x;\epsilon)$ in $C^2(\bar{\Omega})$ so that $z_1 \leq u_i(x;\epsilon) \leq z_3$ for all $x \in \bar{\Omega}$ and they have the following interior transition layers:*

$$\lim_{\epsilon\to 0} u_1(x;\epsilon) = \begin{cases} z_2 & x \in \Omega_1 \\ z_1 & x \in \Omega_2\,, \end{cases}$$

$$\lim_{\epsilon\to 0} u_2(x;\epsilon) = \begin{cases} z_2 & x \in \Omega_1 \\ z_3 & x \in \Omega_2\,, \end{cases}$$

and the above convergence are uniform on every compact subset of indicated regions.

Proof. For any $x \in \Omega_1$ and for any $z_1 < u < z_2$, $f(x,u) > 0$. Since $f(x,z_2) = 0$, $f_u(x,z_2) < 0$, and

$$\int_{z_1}^{z_2} f(x,u)\,du > 0$$

for all $x \in \Omega_1$, by the similar construction on the proof of Theorem 2-2, there is a function $\bar{u}_1(x;\epsilon) \in C^2(\bar{\Omega}_1)$ such that for all sufficiently small $\epsilon > 0$,

$$\begin{aligned}
&\epsilon^2 \Delta \bar{u}_1 + f(x, \bar{u}_1) \geq 0, \quad x \in \Omega_1, \\
&\bar{u}_1(x;\epsilon) = z_1, \quad x \in \partial\Omega_1, \\
&z_1 < \bar{u}_1(x;\epsilon) < z_2, \quad x \in \bar{\Omega}_1,
\end{aligned}$$

and $\lim_{\epsilon \to 0} \bar{u}_1(x;\epsilon) = z_2$ uniformly on every compact subset of Ω_1. Let

$$\bar{w}_1(x;\epsilon) = \begin{cases} \bar{u}_1(x;\epsilon) & \text{if} \quad x \in \Omega_1, \\ z_1 & \text{if} \quad x \in \bar{\Omega} \setminus \Omega_1, \end{cases}$$

Then $\bar{w}_1(x;\epsilon)$ is a quasisubsolution of the problem (I).

Similarly, for any $x \in \Omega_2$ and for any $z_1 < u < z_2$, $f(x,u) < 0$. Since $f(x, z_1) = 0$, $f_u(x, z_1) < 0$, and $\int_{z_1}^{z_2} f(x,u)\,du < 0$ for all $x \in \Omega_2$, there is a function $\hat{u}_1(x;\epsilon) \in C^2(\bar{\Omega}_2)$ such that for all sufficiently small $\epsilon > 0$,

$$\begin{aligned}
\epsilon^2 \Delta \hat{u}_1 + f(x, \hat{u}_1) &\leq 0, \quad && x \in \Omega_2, \\
\hat{u}_1(x;\epsilon) &= z_2, \quad && x \in \partial\Omega_2, \\
z_1 < \hat{u}_1(x;\epsilon) &< z_2, \quad && x \in \bar{\Omega}_2,
\end{aligned}$$

and $\lim_{\epsilon \to 0} \hat{u}_1(x;\epsilon) = z_1$ uniformly on every compact subset of Ω_2. Let

$$\hat{w}_1(x;\epsilon) = \begin{cases} \hat{u}_1(x;\epsilon) & \text{if} \quad x \in \Omega_2, \\ z_2 & \text{if} \quad x \in \bar{\Omega} \setminus \Omega_2 . \end{cases}$$

Then $\hat{w}_1(x;\epsilon)$ is a quasisupersolution of the problem (I). Since

$$\bar{w}_1(x;\epsilon) \leq \hat{w}_1(x;\epsilon)$$

for all $x \in \bar{\Omega}$, there is a solution $u_1(x;\epsilon) \in C^2(\bar{\Omega})$ such that

$$z_1 \leq \bar{w}_1(x;\epsilon) \leq u_1(x;\epsilon) \leq \hat{w}_1(x;\epsilon) \leq z_2$$

for all $x \in \bar{\Omega}$ and for all sufficiently small $\epsilon > 0$.

By the same method, for any $x \in \Omega_1$ and for any $z_2 < u < z_3$, $f(x,u) < 0$. Since $f(x, z_2) = 0$, $f_u(x, z_2) < 0$, and $\int_{z_2}^{z_3} f(x,u)\,du < 0$

for all $x \in \Omega_1$, there is a quasisupersolution $\hat{w}_2(x;\epsilon)$ of the problem (I) such that $\lim_{\epsilon \to 0} \hat{w}_2(x;\epsilon) = z_2$ uniformly on every compact subset of Ω_1 and $\hat{w}_2(x;\epsilon) = z_3$ for all $x \in \bar{\Omega} \setminus \Omega_1$.

Similarly, for any $x \in \Omega_2$ and for any $z_2 < u < z_3$, $f(x,u) > 0$. Since $f(x,z_3) = 0$, $f_u(x,z_3) < 0$, and $\int_{z_2}^{z_3} f(x,u)\,du > 0$ for all $x \in \Omega_2$, there is a quasisubsolution $\bar{w}_2(x;\epsilon)$ of the problem (I) such that $\lim_{\epsilon \to 0} \bar{w}_2(x;\epsilon) = z_3$ uniformly on every compact subset of Ω_2 and $\bar{w}_2(x;\epsilon) = z_2$ for all $x \in \bar{\Omega} \setminus \Omega_2$. We note that $\bar{w}_2(x;\epsilon) \le \hat{w}_2(x;\epsilon)$ for all $x \in \bar{\Omega}$. Hence there is another nontrivial solution $u_2(x;\epsilon)$ of the problem (I) such that

$$z_2 \le \bar{w}_2(x;\epsilon) \le u_2(x;\epsilon) \le \hat{w}_2(x;\epsilon) \le z_3$$

for all $x \in \bar{\Omega}$ and for all sufficiently small $\epsilon > 0$.

Introducing some functions $\bar{h} < h$ and $\hat{h} > h$ which satisfy the conditions on the derivatives at z_1, z_2, and z_3 in Theorem 3-1, we can prove the same existence result in the case of $h'(z_1) \le 0$ and $h'(z_3) \le 0$.

COROLLARY 1. *Theorem 3-1 is true if we replace the assumptions*

$$h'(z_1) < 0 \quad \text{and} \quad h'(z_3) < 0$$

by $h'(z_1) \le 0$ and $h'(z_3) \le 0$, $h(u) < 0$ for all $z_1 < u < z_2$ and $h(u) > 0$ for all $z_2 < u < z_3$.

COROLLARY 2. *In Theorem 3-1, let h have two zeros z_1 and z_2 and no other zeros between them, and let $h'(z_1) < 0$ and $h'(z_2) > 0$. Then there is ϵ_0 such that for all ϵ with $0 < \epsilon \le \epsilon_0$ the problem (I) has a nontrivial solution $u(x;\epsilon)$ in $C^2(\bar{\Omega})$ so that $z_1 < u(x;\epsilon) < z_2$ for all $x \in \bar{\Omega}$,*

$$\lim_{\epsilon \to 0} u(x;\epsilon) = \begin{cases} z_2 & x \in \Omega_1 \\ z_1 & x \in \Omega_2, \end{cases}$$

and the above convergence is uniform on every compact subset of the indicated regions.

From Theorem 3-1, Corollary 1 and 2, we have the following results:

Corollary 3. *Let $f(x,u) = g(x)u(1-u)[k(1-u)+(1-k)u]$. Then there is ϵ_0 such that for all ϵ with $0 < \epsilon \le \epsilon_0$,*

(1) *if either $k > 1$ or $k < 0$, there exist two nontrivial positive solutions $u_i(x;\epsilon)$ in $C^{(}(\bar{\Omega})$ so that $0 < u_i(x;\epsilon) < 1$ for all $x \in \bar{\Omega}$,*

$$\lim_{\epsilon \to 0} u_1(x;\epsilon) = \begin{cases} 0 & x \in \Omega_1 \\ \frac{k}{2k-1} & x \in \Omega_2\,, \end{cases}$$

$$\lim_{\epsilon \to 0} u_2(x;\epsilon) = \begin{cases} 1 & x \in \Omega_1 \\ \frac{k}{2k-1} & x \in \Omega_2\,, \end{cases}$$

(2) *if $k \in \{0, \frac{1}{2}, 1\}$, there exists a nontrivial positive solution $u(x;\epsilon)$ in $C^2(\bar{\Omega})$ so that $0 < u(x;\epsilon) < 1$ for all $x \in \bar{\Omega}$ and*

$$\lim_{\epsilon \to 0} u(x;\epsilon) = \begin{cases} 0 & x \in \Omega_1 \\ 1 & x \in \Omega_2\,, \end{cases}$$

(3) *if $\frac{1}{2} < k < 1$, there exist two nontrivial positive solutions $u_i(x;\epsilon)$ in $C^2(\bar{\Omega})$ so that $0 < u_i(x;\epsilon) < \frac{k}{2k-1}$ for all $x \in \bar{\Omega}$,*

$$\lim_{\epsilon \to 0} u_1(x;\epsilon) = \begin{cases} 0 & x \in \Omega_2 \\ 1 & x \in \Omega_1\,, \end{cases}$$

$$\lim_{\epsilon \to 0} u_2(x;\epsilon) = \begin{cases} 1 & x \in \Omega_2 \\ \frac{k}{2k-1} & x \in \Omega_1\,, \end{cases}$$

(4) *if $0 < k < \frac{1}{2}$, there exist two nontrivial solutions $u_i(x;\epsilon)$ in $C^2(\bar{\Omega})$ so that $\frac{k}{2k-1} < u_i(x;\epsilon) < 1$ for all $x \in \bar{\Omega}$,*

$$\lim_{\epsilon \to 0} u_1(x;\epsilon) = \begin{cases} 1 & x \in \Omega_2 \\ 0 & x \in \Omega_1\,, \end{cases}$$

$$\lim_{\epsilon \to 0} u_2(x;\epsilon) = \begin{cases} \frac{k}{2k-1} & x \in \Omega_2 \\ 0 & x \in \Omega_1\,, \end{cases}$$

and the above convergencies are uniform on every compact subset of indicated regions.

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