J. Korean Math. Soc. **31** (1994), No. 1, pp. 1–9

PROPERTIES OF FINITE GROUPS WHOSE IRREDUCIBLE CHARACTER DEGREES ARE PRIMES

JOONG-SANG SHIN

1. Introduction

Let G be a finite group and let $Irr(G)$ be the set of irreducible complex characters of G. Let $c.d.(G)$ be the set of degrees of all irreducible characters in $Irr(G)$.

I.M.Issacs and D.S.Passman have been obtained a characterization of groups with the property that every nonlinear irreducible character has a prime degree [4]:

THEOREM 1. *Let G be a finite group with the property that every nonlinear irreducible character has prime degree. Suppose further that at least two distinct primes occur. Then there exist primes p and q, $p \neq q$ such that G has one of the following normal series*

(I) $G \underset{p}{>} N \underset{q^2}{>} Z(G) = Z(N)$
with $G/Z(G)$ and N both nonabelian.

(II) $G \underset{p}{>} N \underset{q}{>} A = Z(N) \times R$
with both G/A and N nonabelian and $Z(G) = Z(N)$. Here R is elementary abelian of order r^m for some prime r and N/A acts irreducibly on it. Also $r^m - 1 = q(r^{\frac{m}{p}} - 1)$.

Conversely if group G has either of the above structure then $c.d.(G) = \{1, p, q\}$.

By the above theorem, a group G with the property that every nonlinear irreducible character has a prime degree is precisely a group with $c.d.(G) = \{1, p, q\}$, $p \neq q$ primes.

It is not hard to show that the above Theorem can be restated as follow (cf. [6, chapter II]).

Received August 24, 1992.

THEOREM 2. *Let G be a finite group with c.d.$(G) = \{1, p, q\}$, where p and q are distinct primes. Let K be a maximal element in the set*

$$\{ker\chi | \chi \in Irr(G),\ \chi(1) \neq 1\}$$

and let $N/K = (G/K)'$. Then G/K is a Frobenius group with kernel N/K and $Z(G) = Z(N)$ and one of the following holds.

(I) *$|G : N| = p$, N/K is an elementary abelian q-group of order $q^2, K = Z(G)$, c.d.$(N) = \{1, q\}$ and c.d.$(G/K) = \{1, p\}$*

(II) *$|G : N| = p$, N/K is an elementary abelian q-group of order $q, K = Z(G) \times R$ where R is an elementary abelian q-group of order r^m, c.d.$(N) = \{1, q\}$ and c.d.$(G/K) = \{1, p\}$*

REMARK. (i) In (I) of Theorem 2, if $p > q$ then it must be $p = 3$ and $q = 2$. (ii) The case when $p > q$ does not occur in (II) of Theorem 2.

Throughout this paper we fix the notation in Theorem 2 and we say that G is of type (I) and of type (II) if *c.d.*$(G) = \{1, p, q\}$ and G satisfies the conclusion (I) and (II) in Theorem 2 respectively.

Let G be a finite group and let p and q be primes with $p \neq q$. We denote the matrix of degree type of G by

$$d.t.(G) = \begin{bmatrix} 1 & p & q \\ x & y & z \end{bmatrix}$$

if the following two conditions hold:

(i) *c.d.*$(G) = \{1, p, q\}$

(ii) G has exactly x linear characters, y irreducible characters of degree p and z irreducible characters of degree q.

Our main result is the following.

THEOREM. *Let G be a finite group with c.d.$(G) = \{1, p, q\}$, $p \neq q$ primes. Let s be the order of $Z(G)$. Then the following hold.*

(i) *If G is of type (I) or of type (II) with $r = q$, then the commutator subgroup G' is of order q^3 and*

$$d.t.(G) = \begin{bmatrix} 1 & p & q \\ \frac{ps}{q} & \frac{(q^2-1)s}{pq} & \frac{(q-1)ps}{q} \end{bmatrix}$$

(ii) *If G is of type (II) and $r \neq q$, then G' is of order qr^m and*

$$d.t.(G) = \begin{bmatrix} 1 & p & q \\ ps & \frac{(q-1)s}{p} & \frac{(r^m-1)ps}{q} \end{bmatrix}$$

REMARK. It can be shown that the commutator subgroup G' is isomorphic to the extra special group $M(q)$ of order q^3 if G is of type (I) or of type (II) with $r = q$ and G' is the semidirect product of an elementary abelian group of order r^m and a group of order q.

For notations and terminologies one confer [3].

2. Properties of a finite group G with $c.d.(G) = \{1, p, q\}$

For convenience we descrive the following well known theorems without proof (cf. [3])

THEOREM (CLIFFORD). *Let $N \triangleleft G$ and let $\chi \in Irr(G)$. Let ϕ be an irreducible constituent of χ_N and suppose that $\phi = \phi_1, \cdots, \phi_t$ are the distinct conjugates of ϕ in G. Then $\chi_N = e(\phi_1 + \cdots + \phi_t)$, where $e = [\chi_N, \phi]$ and $t = |G : I_G(\phi)|$.*

THEOREM (ITO). *Let $A \triangleleft G$ be abelian. Then $\chi(1)$ devides $|G : A|$ for all $\chi \in Irr(G)$.*

THEOREM (GALLAGHER). *Let $N \triangleleft G$ and let $\chi \in Irr(G)$ be such that $\chi_N = \xi \in Irr(N)$. Then the characters $\beta\chi$ for $\beta \in Irr(G/N)$ are irreducible, distinct for distinct β and are all of the irreducible constituents of ξ^G.*

Now we investigate some properties of a finite group G with $c.d.(G) = \{1, p, q\}$.

PROPOSITION 2.1. *Suppose that G is of type (I). Let Q be a Sylow q-subgroup of N. Then $G' \subset Q$, $Q \triangleleft G$ and $N' = G''$ is of order q.*

Proof. Let P be a Sylow p-subgroup of G. Since $G/Z(G) = G/K$ is of order pq^2, the factor group $P/Z(P)$ is a cyclic group and so P is abelian. It follows [3, cor 12. 34] that every Sylow s-subgroup of G is abelian normal in G for all primes $s \neq p, q$. Thus we have $|G' \cap Z(G)|$ is a power of q (cf. [3, theorem 5.6]).

Since $N/K = (GK)'$ is of order q^2, we have

$$q^2|Z(G)| = |N| = \frac{|G'||Z(G)|}{G' \cap Z(G)|}$$

and so $|G'| = q^2|G' \cap Z(G)|$. Thus G' is a q-group.

Since $c.d.(N) = \{1, q\}$, the subgroup N has a normal q-complement L (cf. [3, cor 12.2]). Since $|N : Z(N)| = q^2$, we have $L \subset Z(N)$. Now it follows that $Q \triangleleft N$ since $N = QL$. Moreover, Q is also a Sylow q-subgroup of G and a characteristic subgroup of N. Thus $Q \triangleleft G$ and $G' \subset Q$.

Let B be a normal subgroup of N with $|B : K| = q$. Then $B/Z(B)$ is cyclic and hence B is abelian. Now it follows from [3, lemma 12.12] that

$$|B| = |N'||Z(N)| = |N'||K|$$

and so $|N'| = q$.

Since $N = G'Z(G) = QZ(G)$, we have $N' = G'' = Q'$. Thus the proposition holds.

PROPOSITION 2.2. *Suppose that G is of type (II) and $r \neq q$. Then $G'' = N'$ and N' is an elementary abelian group of order r^m.*

Proof. Let Q be a Sylow q-subgroup of N. Since $K = Z(G) \times R$ for some elementary abelian group R of order r^m, it follows that $G/Z(G)$ is of order pqr^m and so $|Q/Z(Q)| \leq q$. Thus Q is abelian.

Since $c.d.(N) = \{1, q\}$, every Sylow subgroup of N is abelian (cf. [3, cor 12.34]). By Theorem 5.6 of [3], we have

$$N' \cap Z(N) = \{1\}.$$

Note that K is abelian and N/K is cyclic of order q. Thus we have the following (cf. [3, Lemma 12.12])

$$|K| = |N'||K \cap Z(N)| = |N'||Z(N)|.$$

Since $N'Z(N) \subset K$, we have $K = N' \times Z(N)$.

Now it follows from $K = R \times Z(N)$ that $N' \cong R$ and so N' is an elementary abelian group of order r^m.

Note that $N = G'K$ and $N' \subset G'$. Thus we have

$$N = G'K = G'N'Z(N) = G'Z(G)$$

and so $N' = G''$.

PROPOSITION 2.3. *Suppose that G is a finite group with c.d.$(G) = \{1, p, q\}$. Then the set of all irreducible characters of G of degree q is precisely the set of all irreducible constituents of ϕ^G, where ϕ runs over all nonlinear irreducible characters of G'.*

Proof. Let ϕ be an irreducible character of G' of degree q. If χ be an irreducible constituent of ϕ^G, then Clifford Theorem yields that $\chi(1) = q$.

Now we show that every irreducible character of G of degree q is an irreducible constituent of ϕ^G for some $\phi \in \mathrm{Irr}(G')$ of degree q.

Let χ be an irreducible character of G of degree q. Assume that χ_N has a linear constituent θ. Then Gallagher Theorem yield that θ^G has no linear constituent since χ is an irreducible constituent of θ^G. Moreover, θ^G has no irreducible constituent of degree p since $\theta^G(1) = p$ and χ is an irreducible constituent of θ^G. Thus every irreducible constituent of θ^G is of degree q. Since p and q are distinct primes, this is not the case. Thus χ_N is an irreducible character of degree q and so $N' \not\subseteq \ker \chi_N$.

Now let ϕ be an irreducible constituent of $\chi_{G'}$. Then, by Clifford Theorem, we have

$$\chi_{G'} = e(\phi_1 + \phi_2 + \cdots + \phi_t)$$

, where $\phi_1, \cdots, \phi_t$ are the distinct conjugates of ϕ in G and $e = [\phi, \chi_{G'}]$. Since

$$\bigcap_{i=1}^{t} \ker \phi_i = \ker \chi_{G'} = G' \cap \ker \chi_N \not\supseteq N'$$

, we have $N' \not\subseteq \ker \phi_i$ for some i.

Since $N' = G''$ by Proposition 2.1 and Proposition 2.2, the character ϕ_i is not linear. This implies that $\phi(1) = q$ and so $\chi_{G'} = \phi \in \mathrm{Irr}(G')$. Thus χ is an irreducible constituent of ϕ^G. This completes the proof.

Let G be a finite group. If $H \lhd G$ and χ is a character of G with $H \subset \ker \chi$, then there is a unique character $\bar{\chi}$ of G/H defined by $\bar{\chi}(Hg) = \chi(g)$. This formula can also be used to define the character χ if $\bar{\chi}$ is given. It is immediate consequence that χ is irreducible if and only if $\bar{\chi}$ is. In this paper, we will not distinguish between χ and $\bar{\chi}$.

3. Main Result

In this section we will prove our main Theorem.

THEOREM 3.1. *Suppose that G is of type (I) or of type (II) with $r = q$. Let $s = |Z(G)|$. Then the commutator subgroup G' is of order q^3 and*

$$d.t.(G) = \begin{bmatrix} 1 & p & q \\ \frac{ps}{q} & \frac{(q^2-1)s}{pq} & \frac{(q-1)ps}{q} \end{bmatrix}$$

Proof. Let $\phi \in \mathrm{Irr}(G)$. If $\phi(1) = 1$, then $\phi_K \in \mathrm{Irr}(K)$ and

$$N' \subset K \cap G' \subset \ker \phi_K.$$

Thus $\phi_K \in \mathrm{Irr}(K/N')$.

Now suppose that $\phi(1) = p$. Since $K = Z(G)$, every irreducible constituent of ϕ_K is invariant in G. Thus it follows by Clifford Theorem that $\phi_K = p\theta$ for some $\theta \in \mathrm{Irr}(K)$. Since $c.d.(N) = \{1, q\}$ and $|G : N| = p$, we have

$$\phi_N = \chi_1 + \chi_2 + \cdots + \chi_p$$

for some linear characters $\chi_1, \chi_2, \cdots, \chi_p$ of N.

Since $N' \subset K \cap \ker \chi_i = \ker (\chi_i)_K$ for all $i = 1, 2, \cdots, p$, we have

$$N' \subset \ker \phi_K.$$

Since $\phi_K = p\theta$, it follows that $\theta \in \mathrm{Irr}(K/N')$ and ϕ is an irreducible constituent of θ^G.

In the proof of Proposition 2.3, we showed that if ϕ is an irreducible character of G of degree q then $\phi_N \in \mathrm{Irr}(N)$ and $N' \not\subseteq \ker \phi_N$. In this case we have $N' \not\subseteq \ker \phi_K$ since $N' \subset K$.

Now we can conclude that $\theta \in \mathrm{Irr}(K/N')$ is extendible to G and every irreducible constituent of $\theta \in \mathrm{Irr}(K) - \mathrm{Irr}(K/N')$ is of degree q.

Since K/N' is abelian and G/K has p linear characters and $\frac{(q^2-1)}{p}$ irreducible characters of degree p, it follows by Gallagher Theorem that G has $p|K : N'|$ linear characters and $|K : N'|\frac{(q^2-1)}{p}$ irreducible characters of degree p.

If $\phi \in \operatorname{Irr}(G)$ is of degree q, then $\phi_K = q\theta$ for some $\theta \in \operatorname{Irr}(K) - \operatorname{Irr}(K/N')$. In this case, θ^G has p irreducible constituent of degree q. Thus G has $p(|K| - |K : N'|)$ irreducible characters of degree q.

Note that G has $|G : G'|$ linear characters. Thus $|G : G'| = p|K : N'|$ and then

$$\begin{aligned} p|K : K \cap G'| &= p|KG' : G'| = |G : G'| = p|K : N'| \\ &= p|K : K \cap G'||K \cap G' : N'| \end{aligned}$$

Thus $|K \cap G' : N'| = 1$. That is, $K \cap G' = N'$.

Since $|N : K| = q^2$ and $|N'| = q$, we have $|G' : K \cap G'| = |N : K| = q^2$ and

$$|G'| = |G' : K \cap G'||K \cap G'| = q^2|N'| = q^3.$$

REMARK. If G is of type (II) with $r = q$, then $|N : Z(G)| = q^2$ (cf. [4]). The proof of this case is similar to type(I).

THEOREM 3.2. *Suppose that G is of type (I) and $r \neq q$. Let $s = |Z(G)|$. Then the commutator subgroup G' is of order qr^m and*

$$d.t.(G) = \begin{bmatrix} 1 & p & q \\ ps & \frac{(q-1)s}{p} & \frac{(r^m-1)ps}{q} \end{bmatrix}$$

Proof. Let θ be an irreducible character of $Z(G)$. Assume that θ^N has x linear constituents and y irreducible constituents of degree q. Then since θ is invariant in G and $c.d.(N) = \{1, q\}$, we have the equation

$$qr^m = \theta^N(1) = x + yq^2.$$

Note that $q|(r^m - 1)$ (cf. Theorem 1). Thus $x \neq 0$ and so θ is extendible to N. Since $(N/Z(G))' = N'Z(G)/Z(G) = K/Z(G)$, θ^N has $\frac{(r^m-1)}{q}$ irreducible constituents of degree q by Gallagher Theorem.

Note that every irreducible character of N of degree q is extendible to G (cf. Proof of Proposition 2.3).

Since $(\theta^N)^G = \theta^G$, Gallagher Theorem yields that θ^G has $\frac{p(r^m-1)}{q}$ irreducible constituents of degree q.

Assume that θ is not extendible to G. Then since θ is invariant in G, we have the equation

$$pqr^m = \theta^G(1) = xp^2 + yq^2$$

, where x and y are the numbers of irreducible constituents of degree p and of degree q respectively. But since $y = \frac{p(r^m-1)}{q}$ and $p \nmid q$, it is not the case.

Thus every $\theta \in \operatorname{Irr}(Z(G))$ is extendible to G.

Now, by applying Proposition 2.3 to $G/Z(G)$, it follows that $G/Z(G)$ has p linear characters, $\frac{(q-1)}{p}$ irreducible characters of degree p and $\frac{(r^m-1)p}{q}$ irreducible characters of degree q. Thus, by Gallagher Theorem, G has $p|Z(G)|$ linear characters, $\frac{(q-1)|Z(G)|}{p}$ irreducible characters of degree p and $\frac{(r^m-1)p|Z(G)|}{q}$ irreducible characters of degree q.

Finally we show that $G'$is of order qr^m.

Note that G has $|G : G'|$ linear characters and that

$$N/G' = G'Z(G)/G' \cong Z(G)/G' \cap Z(G).$$

Thus we have

$$p|Z(G)| = |G : G'| = p|Z(G) : G' \cap Z(G)|$$

and so $G' \cap Z(G) = \{1\}$.

Since $G'Z(G)/Z(G) \cong G'/G' \cap Z(G)$, we have

$$\begin{aligned} |G'| &= |G' : G' \cap Z(G)||G' \cap Z(G)| \\ &= |G'Z(G) : Z(G)| \\ &= |N : Z(G)| \\ &= qr^m. \end{aligned}$$

We have proved THEOREM which is introduced in section 1.

References

1. Dornhoff, L., *Group representation theory*, part I, II, Marcel Dekker, New York, 1971, 1972.
2. Gorenstein, D., *Finite groups*, Chelsea, New York, 1980.
3. Issacs, I. M., *Character theory of finite groups*, Academic Press, New York, 1976.
4. Issacs, I. M. and Passman, D. S., *A characterization of groups in terms of degrees of their characters I, II*, Pacific J. Math. **24** (1968), 467-510.

5. Issacs, I. M. and Passman, D. S., *Finite groups with small character degrees and large prime divisors I, II*, Pacific J. Math. **29** (1969), 311-324.
6. Shin, J. S., *A characterization of finite groups whose degrees of irreducible characters are 1, 2 and 3*, Ph. D. Thesis, So-Gang University, 1987.
7. Suzuki, M., *Group theory I, II*, Springer-Verlag, New York, 1982, 1986.

Department of Mathematics
Kyung-won University
Sung-nam 461-200, Korea