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FIXED POINTS FOR SET–VALUED INCREASING OPERATORS AND APPLICATIONS

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I. Introduction and Preliminaries

The existence of fixed points for set-valued increasing operators is one of important problems in the study of nonlinear analysis [4], [6]. In this paper, we give two fixed point theorems for nonlinear set-valued increasing operators by using the generalized Gwinner's theorem [7] and the generalized KKM theorem [3] on H-spaces. As an application, we prove a basic theorem which is important in the mathematical economies.

Let $\mathcal{F}(X)$ be the family of all nonempty finite subsets of X.

DEFINITION 1 ([2]). (1) Let X be a topological space and $\{\Gamma_A\}$ be a family of nonempty contractible subsets of X, indexed by $A \in \mathcal{F}(X)$, such that $A \subset B$ implies $\Gamma_A \subset \Gamma_B$ for $A, B \in \mathcal{F}(X)$. The pair $(X, \{\Gamma_A\})$ is called an *H-space.*

(2) Let $(X, \{\Gamma_A\})$ be an H-space. A subset D of X is said to be *H-convex* if for every finite subset A of $D, \Gamma_A \subset D$.

DEFINITION 2 ([3]). Let X be a nonempty set and $(Y, \{\Gamma_A\})$ be an H-space. A set-valued mapping $F : X \to 2^Y$ is called a *generalized KKM mapping* if for any finite set $\{x_1, x_2, \ldots, x_n\}$ in X, there exists a finite set $\{y_1, y_2, \ldots, y_n\}$ in Y such that for any subset $\{y_{i_1}, y_{i_2}, \ldots, y_{i_k}\} \subset \{y_1, y_2, \ldots, y_n\}, 1 \leq k \leq n, \Gamma_{\{y_{i_1}, y_{i_2}, \ldots, y_{i_k}\}} \subset \bigcup_{j=i}^{k} F(x_{i_j})$.

We say that a subset C of a topological space X is compactly closed (resp., compactly open) in X if, for every compact set K in X, the set $C \cap K$ is closed (resp., open) in K.

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THEOREM 1 ([3]). *(The Generalized KKM Theorem)*
Let X be a nonempty set, $(Y, \{\Gamma_A\})$ be an H-space and $F : X \to 2^Y$ be a generalized KKM mapping satisfying one the following conditions:
(1) *for each $x \in X, F(x)$ is compactly closed in Y,*
(2) *for each $x \in X, F(x)$ is compactly open in Y.*
Then the family $\{F(x) : x \in X\}$ of sets has the finite intersection property. In addition, if there exists an $x_o \in X$ such that $F(x_o)$ is a compact set, then $\bigcap_{x\in X} F(x) \neq \phi$.

THEOREM 2 ([7]). *(The Generalized Gwinner's Theorem)*
Let $(X, \{\Gamma_A\})$ be an H-space, E be closed subset of X and $G : E \to 2^X$ be a generalized KKM mapping satisfying the following conditions :
(1) *for each $x \in E, G(x)$ is a finite closed subset in X (i.e. the intersecion of $G(x)$ and any finite subset L of X is closed in the euclidean topology),*
(2) *there exists an $x_0 \in E$ such that the closure, $\overline{G(x_0)}$, of $G(x_0)$ is a compact subset in X,*
(3) *for any finite set $D = E \cap F$ containing x_0, where F is a finite subset of X containing $x_0, \overline{\bigcap_{y\in D} G(y)} \cap D = \bigcap_{y\in D} G(y) \cap D$.*
Then $\bigcap\{G(x) : x \in E\} \neq \phi$.

DEFINITION 3 ([8]). Let X be a Hausdorff topological space with the partially ordered structure and D be a subset of X. The set D is said to be *upper semi-closed* in X if for any directed sequence $\{x_\alpha : \alpha \in I\}$ in D net-convering to $\bar{x}$, $x_\alpha \leq \bar{x}$ for all $\alpha \in I$, we have $\bar{x} \in D$.

It is easy to show that any closed set in X is upper semi-closed.

DEFINITION 4 ([8]). Let X, Y be partially ordered sets and M be a subset of X. A set-valued operator $A : M \to 2^Y$ is said to be *increasing* if, for any $x, y \in M, x \leq y$ and $u \in Ax$, there exists a $v \in Ay$ such that $u \leq v$.

If A is a single-valued operator, then A is increasing operator in the sense of Definition 4 if and only if $x \leq y$ implies $Ax \leq Ay$.

II. Fixed Point Theorems

Now, in this section, we give our main theorems for set-valued increasing operators.

LEMMA 1. *Let $(X, \{\Gamma_A\})$ be an ordered H-space and D be a upper semi-closed subset in X. Suppose further that*

(1) *the space X satisfies the consistent axiom, i.e., for any two directed sequences $\{x_\alpha : \alpha \in I\}$ and $\{y_\alpha : \alpha \in I\}$, if $x_\alpha \leq y_\alpha$ for all $\alpha \in I$, $\{x_\alpha\}$ net-converges to $\bar{x}$, and $\{y_\alpha\}$ net-converges to $\bar{y}$, then we have $\bar{x} \leq \bar{y}$,*

(2) *$\Omega(\widetilde{y}) = \{y \in D : y \not\leq \widetilde{y}\}, \widetilde{y} \in X$, is an H-convex set,*

(3) *for any totally ordered subset N of D, there exist a compact set K in X and $x_o \in K \cap \overline{N}$ such that $x_o \not\leq x$ for all $x \in \overline{N} \setminus K$.*

Then D has a maximal element.

Proof. For any $a \in D$, put $D(a) = \{y \in D : a \leq y\}$. It is sufficient to show that $D(a)$ has a maximal element. Choose a totally ordered set N in $D(a)$ and let $\overline{N}$ denote the closure of N in X. Letting $B(x) = \{y \in \overline{N} : x \leq y\}, x \in N$. Then $B(x) = \overline{N} \cap \{y \in X : x \leq y\}, x \in N$, is a closed set in $D(a)$ and so $B(x)$ is a finite closed set and $\bigcap_{x \in N} B(x)$ is closed. Therefore, we have

$$\overline{\bigcap_{x \in N} B(x)} \cap N = \bigcap_{x \in N} B(x) \cap N.$$

Now, we prove that $B : N \to 2^X$ is a generalized KKM mapping. Suppose that B is not a generalized KKM mapping. Then there exists a finite subset $A = \{y_1, y_2, \dots, y_n\}$ of N such that $\Gamma_A \not\subset \bigcup_{i=1}^n B(y_i)$. Hence there exists a point $\widetilde{x} \in \Gamma_A$ such that $\widetilde{x} \notin \bigcup_{i=1}^n B(y_i)$, i.e., $\widetilde{x} \notin B(y_i), i = 1, 2, \dots, n$, which implies $y_i \not\leq \widetilde{x}, i = 1, 2, \dots, n$. Thus we have $\{y_1, y_2, \dots, y_n\} \subset \Omega(\widetilde{x})$. Since $\Omega(\widetilde{x})$ is H-convex, $\widetilde{x} \in \Gamma_A \subset \Omega(\widetilde{x})$ and so $\widetilde{x} \not\leq \widetilde{x}$, which is a contradiction.

From the condition (3), for each $x \in \overline{N} \setminus K$, we have $x \notin B(x_o)$, i.e., $B(x_o) \subset K$. On the other hand, since $B(x_o) \subset N, B(x_o) \subset \overline{N} \cap K \subset K$ and the closure, $\overline{B(x_o)}$, of $B(x_o)$ is compact. Thus, from the arguments above, we know that B satisfies all the conditions of the generalized Gwinner's theorem and so $\bigcap_{x \in N} B(x) \neq \phi$. Take $\bar{y} \in \bigcap_{x \in N} B(x)$. Then, from the definition of B, we have $y \leq \bar{y}$ for all $y \in N$. On the other hand, since $\bar{y} \in \bigcap_{x \in N} B(x) \subset \overline{N}$, there exists a directed sequence $\{y_\alpha : \alpha \in I\} \subset N \subset D(a)$ such that $\{y_\alpha\}$ net-converges to $\bar{y}$. Since $y_\alpha \leq \bar{y}$ for all $\alpha \in I$ and D is upper semi-closed in X, we have $\bar{y} \in D$. Since $N \subset D(a)$, we have $a \leq y$ for all $y \in N$. Hence, from $y \leq \bar{y}$, we have $a \leq \bar{y}$ and so $\bar{y} \in D(a)$. Therefore, $\bar{y}$ is the supremum of N in $D(a)$ and so, by Zorn's lemma, $D(a)$ has a maximal element. This completes the proof.

THEOREM 3. *Let $(X, \{\Gamma_A\})$ be an ordered H-space satisfying the consistent axiom, M be a closed subset of X and $A : M \to 2^M$ be a set-valued increasing opereator. Suppose further that*

(1) $D = \{x \in M : \exists u \in Ax$ such that $x \leq u\}$ is nonempty,

(2) $\Omega(\widetilde{y}) = \{y \in D : y \not\leq \widetilde{y}\}, \widetilde{y} \in X$, is an H-convex set,

(3) for each totally ordered subset N of D, there exist a compact set K in X and $x_o \in K \cap \overline{N}$ such that $x_o \not\leq x$ for all $x \in \overline{N} \setminus K$,

(4) for any $x \in M$, Ax is a compact subset of X.

Then A has a fixed point in M.

Proof. First, we prove that D is an upper semi-closed set. Let $\{x_\alpha : \alpha \in I\}$ be a directed sequence in D which net-converges to $\bar{x}$ and $x_\alpha \leq \bar{x}$ for all $\alpha \in I$. For all $\alpha \in I$, since $x_\alpha \in D$, there exists a $u_\alpha \in Ax_\alpha$ such that $x_\alpha \leq u_\alpha$. Thus, since A is a set-valued increasing operator, from $x_\alpha \leq \bar{x}$ and $x_\alpha \leq u_\alpha$ for all $\alpha \in I$, there exists a $y_\alpha \in A\bar{x}$ such that $u_\alpha \leq y_\alpha$ and so $x_\alpha \leq y_\alpha$ for all $\alpha \in I$. From the condition (4), since $A\bar{x}$ is compact, $\{y_\alpha : \alpha \in I\}$ have a directed subsequence $\{y_\tau : \tau \in \Lambda\}, \Lambda \subset I$, net-converging to a point $\bar{y} \in A\bar{x}$ (See Theorem 2 of Chapter 5 [5]). Since the directed subsequence $\{x_\tau : \tau \in \Lambda\}$ of $\{x_\alpha : \alpha \in I\}$ net-converges also to the point $\bar{x}$ and $x_\tau \leq y_\tau$ for all $\tau \in \Lambda$, by the condition (1) of Lemma 1, we have $\bar{x} \leq \bar{y}$ and so, since $\bar{y} \in A\bar{x}, \bar{x} \in D$. This implies that D is upper semi-closed. Thus, from (4) and Lemma 1, D has a maximal element x^* and hence there exists a $u^* \in Ax^*$ such that $x^* \leq u^*$. Since A is a set-valued increasing operator, there exists a $y^* \in Au^*$ such that $u^* \leq y^*$, which implies $u^* \in D$. Since $x^* \leq u^*$ and x^* is a maximal element, we have $x^* = u^* \in Ax^*$. This completes the proof.

LEMMA 2. *Let $(X, \{\Gamma_A\}), D, \Omega(\widetilde{y}), N$ be the same as in Lemma 1. If there exists an $x_o \in N$ such that $G(x_o) = \{y \in \overline{N} : x_o \leq y\}$ is a relatively compact set in X, i.e., the closure, $\overline{G(x_o)}$, of $G(x_o)$ is a compact set, then D has a maximal element.*

Proof. As in the proof of Lemma 1, $G : N \to 2^X$ is a generalized KKM mapping and, for any $x \in N$, $G(x)$ is closed in $D(a)$. Thus, by the assumption, there exists an $x_o \in N$ such that $G(x_o)$ is relatively compact in X and so $G(x_o)$ is compact. Therefore, by Theorem 1, $\bigcap_{x\in N} G(x) \neq \phi$. This completes the proof.

From Lemma 2, we have the following:

THEOREM 4. *Let* $(X, \{\Gamma_A\}), M, A$ *satisfy the conditions* (1), (2), (4) *in Theorem 3 and the following condition (3)':*

(3)' *for any totally ordered subset* N *of* D, *there exists an* $x_o \in N$ *such that* $G(x_o) = \{y \in \overline{N} : x_o \leq y\}$ *is a relatively compact set in* X. *Then* A *has a fixed point in* M.

III. Application

In this section, as an application, by using Theorem 4, we show the existence of equilibrium points of mathematical economics:

THEOREM 5. *Let* (X, Γ_A) *and* (Y, Γ_B) *be ordered H-spaces satisfying the consistent axiom,* L *be a nonempty closed subset of* Y *and* K *be a compact set in* X. *Suppose further that the sets* $\Omega(\widetilde{x}) = \{x \in K : x \not\leq \widetilde{x}\}$, $\widetilde{x} \in \Gamma_A$, and $\Omega(\widetilde{y}) = \{y \in L : y \not\leq \widetilde{y}\}$, $\widetilde{y} \in \Gamma_B$, are *H-convex. If the following conditions are satisfied:*

(1) $T : K \to 2^L$ *is a set-valued increasing operator and, for any* $x \in K, Tx$ *is compact,*

(2) $\varphi : K \times L \to R$ *is a continuous mapping and there exists a constant* C *such that* $\varphi(x, y) \geq C$ *for any* $x \in K$ *and* $y \in Tx$,

(3) *for any* $y_i \in L, i = 1, 2, y_1 \leq y_2$ *and for any* $\xi_1 \in K$ *such that* $\varphi(\xi_1, y_1) \leq \varphi(x, y_1)$ *for all* $x \in K$, *if there exists a* $\xi_2 \in L$ *such that* $\varphi(\xi_2, y_2) \leq \varphi(x, y_2)$ *for all* $x \in K$, *than we have* $\xi_1 \leq \xi_2$,

(4) *for any tatally ordered subsets* N_K *and* N_L *of* K *and* L, *respectively, there exist* $x_o \in N_K$ *and* $y_o \in N_L$ *such that the sets* $G_K(x_o) = \{x_o \in \overline{N_K} : x_o \leq x\}$ *and* $G_L(x_o) = \{y \in \overline{N_L} : y_o \leq y\}$ *are relatively compact in* X *and* Y, *respectively,*

(5) *there exist* $x_o, x_1 \in K$ *and* $y_o, y_1 \in L$ *such that if* $y_1 \in Tx_o$ *and* $\varphi(x_1, y_o) \leq \varphi(x, y_o)$ *for all* $x \in K$, *then we have* $x_o \leq x_1$ *and* $y_o \leq y_1$, *then there exist* $\bar{x} \in K$ *and* $\bar{y} \in T(\bar{x})$ *such that* $C \leq \varphi(x, \bar{y})$ *for all* $x \in K$.

Proof. Define a mapping $S : L \to 2^K$ by

$$S(y) = \{x' \in K : \varphi(x', y) \leq \varphi(x, y) \text{ for all } x \in K\}.$$

Since φ is continuous and K is compact, $S(y)$ is a nonempty closed set and so $S(y)$ is compact. From the condition (3), it follows that S is a set-valued increasing operator.

On the other hand, define a mapping $R : K \times L \to 2^{K\times L}$ by $R(x,y) = S(y) \times T(x)$ and define a partial order on $Z = X \times Y$ by $u_1 \leq u_2$ for $u_1 = (x_1, y_1), u_2 = (x_2, y_2) \in Z$ if and only if $x_1 \leq x_2$ and $y_1 \leq y_2$. Then $(Z, \{\Gamma_{A\times B}\})$ is a partially ordered H-space satisfying the consistent axiom, where $\Gamma_{A\times B} = \Gamma_A \times \Gamma_B$, and $M = K \times L$ is a closed set in Z. It is easy to see that

$$D_L = \{y \in L : \exists \eta \in T(x) \text{ such that } y \leq \eta\} \neq \phi.$$

Since $S(y) \neq \phi$,

$$D_K = \{x \in K : \exists \xi \in S(y) \text{ such that } x \leq \xi\} \neq \phi$$

and

$$D = \{v \in M : \exists \mu \in Rv \text{ such that } v \leq \mu\} = D_K \times D_L \neq \phi.$$

Putting $\widetilde{\mu} = (\widetilde{x}, \widetilde{y})$, by the assumptions, for $\widetilde{\mu} \in \Gamma_{A\times B}$,

$$\Omega(\widetilde{\mu}) = \{\mu \in D : \mu \leq \widetilde{\mu}\} = \Omega(\widetilde{x}) \times \Omega(\widetilde{y})$$

is H-convex. Since S and T are increasing, R is also increasing. Since, for any totally ordered subset $N = N_K \times N_L$ of D, there exists a $u_o = (x_o, y_o) \in N$ such that

$$G(u_o) = \{(x,y) \in \overline{N} : (x_o, y_o) \leq (x,y)\} \subset G_K(x_o) \times G_L(y_o),$$

by (4), $G(u_o)$ is relatively compact is Z. Since $T(y)$ and $T(x)$ are compact, for any $u = (x,y) \in M$, Ru is also compact. It is easy to show that the condition (5) holds if and only if there exist $x_o \in K, y_o \in L$ and $x_1 \in S(y_o), y_1 \in T(x_o)$ such that $x_o \leq x_1$ and $y_o \leq y_1$ if and only if there exist $u_o = (x_o, y_o) \in M$ and $u_1 = (x_1, y_1) \in Ru_o$ such that $u_o \leq u_1$. Thus, by Theorem 4, there exists a $(\bar{x}, \bar{y}) \in K \times L$ such that $(\bar{x}, \bar{y}) \in S(\bar{y}) \times T(\bar{x})$, i.e., $\bar{x} \in S(\bar{y}) \subset K$ and $\bar{y} \in T(\bar{x}) \subset L$. Therefore, from the definition of S and (2), we have $C \leq \varphi(\bar{x}, \bar{y}) \leq \varphi(x, \bar{y})$ for all $x \in K$. This completes the proof.

REMARK. Theorem 5 improves the Walras theorem ([1]). Recently, some authors [8], [10]-[12] gave some relations between fixed point Theorems and the existence of equilibrium points of abstract economies.

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