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A NOTE ON STARLIKENESS OF A CERTAIN INTEGRAL

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Let A be the class of functions $f(z)$ which are analytic in the open unit disk U with the normalizations $f(0) = 0$ and $f'(0) = 1$. Denoting by $R(\alpha)$ the subclass of A consisting of functions $f(z)$ which satisfy $\mathrm{Re}\{f'(z)\} > \alpha$ for some $\alpha(\alpha < 1)$ and for all $z \in U$, the starlikeness of an integral $g(z) = \int_0^z \{f(t)/t\}dt$ is shown.

1. Introduction

Let A denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the open unit disk $U = \{z : |z| < 1\}$. A function $f(z)$ belonging to A is said to be a member of the class $R(\alpha)$ if it satisfies

$$\mathrm{Re}\{f'(z)\} > \alpha \qquad (z \in U) \tag{1.2}$$

for some $\alpha(\alpha < i)$. Further, a function $f(z) \in A$ is said to be in the class $S^*(\beta)$ if it satisfies

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \beta \qquad (z \in U) \tag{1.3}$$

for some $\beta(\beta < 1)$.

For $f(z)$ belonging to A, we define the function $g(z)$ defined by the following integral

$$g(z) = \int_0^z \frac{f(t)}{t} dt. \tag{1.4}$$

For such an integral, Singh and Singh [6] have shown

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THEOREM A. *If* $f(z) \in R(0)$, *then* $g(z) \in S^*(0)$.

In the present paper, we improve the above theorem by Singh and Singh [6]. Furthermore, Bulboaca [1, p. 162] has given

PROBLEM. *If* $f(z) \in R(\alpha)$, *find the best* $Q(\alpha)$ *for which* $g(z) \in S^*(Q(\alpha))$; *or for a given* α, *find the best* $\Psi(\alpha)$ *for which* $f(z) \in R(\Psi(\alpha))$ *implies* $g(z) \in S^*(\alpha)$.

2. Starlikeness of the integral

We begin with the statement of the following lemma due to Owa, Ma and Liu [4, Corollary 1].

LEMMA 1. *If* $f(z) \in R(\alpha)$, *then*

$$Re\left\{\frac{f(z)}{z}\right\} > 2\alpha - 1 + 2(1-\alpha)\log 2 \qquad (z \in U). \tag{2.1}$$

The result is sharp.

Further, we have to recall here the following lemma by Jack [2] (also, by Miller and Mocanu [3]).

LEMMA 2. *Let* $w(z)$ *be regular in* U, *with* $w(0) = 0$. *If* $|w(z)|$ *attains its maximum value in the circle* $|z| = r < 1$ *at a point* $z_0 \in U$, *then*

$$z_0 w'(z_0) = k w(z_0), \tag{2.2}$$

where k *is real and* $k \geq 1$.

An application of the above lemmas derives

THEOREM 1. *If* $f(z) \in R(\alpha)$ *with* $\gamma \leq \alpha < 1$, *then* $g(z) \in S^*(\beta)$, *where* $0 \leq \beta \leq \frac{1}{2}$, $t = 2\beta^2 + \beta - 1$, *and*

$$\gamma = \frac{8t\log 2 - 4t(\log 2)^2 - 3t}{8t\log 2 - 4t(\log 2)^2 - 4t + 2}. \tag{2.3}$$

Proof. Since

$$\mathrm{Re}\{f'(z)\} = \mathrm{Re}\{g'(z) + zg''(z)\} > \alpha, \tag{2.4}$$

Lemma 1 gives that

$$\operatorname{Re}\left\{\frac{f(z)}{z}\right\} = \operatorname{Re}\{g'(z)\} > 2\alpha - 1 + 2(1-\alpha)\log 2, \tag{2.5}$$

so that,

$$\operatorname{Re}\left\{\frac{g(z)}{z}\right\} > 4\alpha - 3 + 8(1-\alpha)\log 2 - 4(1-\alpha)(\log 2)^2. \tag{2.6}$$

Define the function $w(z)$ by

$$\frac{zg'(z)}{g(z)} = \beta + (1-\beta)\frac{1+w(z)}{1-w(z)} \qquad (w(z) \neq 1). \tag{2.7}$$

Then $w(z)$ is regular in U and $w(0) = 0$. It is easy to see that

$$\begin{aligned} &\operatorname{Re}\{f'(z)\} \\ =&\operatorname{Re}\{g'(z) + zg''(z)\} \\ =&\operatorname{Re}\left\{\frac{g(z)}{z}\left(\left(\beta + (1-\beta)\frac{1+w(z)}{1-w(z)}\right)^2 + (1-\beta)\frac{2zw'(z)}{(1-w(z))^2}\right)\right\} \end{aligned} \tag{2.8}$$

If we suppose that there exists a point $z_0 \in U$ such that

$$\max_{|z| \leq |z_0|} |w(z)| = |w(z_0)| = 1 \qquad (w(z_0) \neq 1),$$

then we can write $w(z_0) = e^{i\theta}$ $(0 \leq \theta < 2\pi)$. Therefore, applying Lemma 2, we have

$$\begin{aligned} &\operatorname{Re}\{f'(z_0)\} \\ =&\operatorname{Re}\left\{\frac{g(z_0)}{z_0}\left(\beta + (1-\beta)\frac{1+w(z_0)}{1-w(z_0)} + (1-\beta)\frac{2kw(z_0)}{(1-w(z_0))^2}\right)\right\} \\ =&\left(\beta^2 + (1-\beta)^2\frac{\cos\theta + 1}{\cos\theta - 1} + \frac{k(1-\beta)}{\cos\theta - 1}\right)\operatorname{Re}\left\{\frac{g(z_0)}{z_0}\right\} \\ \leq&\left(\beta^2 - \frac{k(1-\beta)}{2}\right)\operatorname{Re}\left\{\frac{g(z_0)}{z_0}\right\} \\ \leq&\left(\beta^2 - \frac{(1-\beta)}{2}\right)\operatorname{Re}\left\{\frac{g(z_0)}{z_0}\right\} \\ \leq&(2\beta^2 + \beta - 1)\left\{2\alpha - \frac{3}{2} + 4(1-\alpha)\log 2 - 2(1-\alpha)(\log 2)^2\right\}, \end{aligned} \tag{2.9}$$

because

$$\operatorname{Re}\left\{\frac{g(z_0)}{z_0}\right\} > 4\alpha - 3 + 8(1-\alpha)\log 2 - 4(1-\alpha)(\log 2)^2 > 0$$

for $\gamma \leq \alpha < 1$, where γ is the root of the equation

$$(2\beta^2 + \beta - 1)\left\{2\gamma - \frac{3}{2} + 4(1-\gamma)\log 2 - 2(1-\gamma)(\log 2)^2\right\} = \gamma.$$

Further, noting that

$$\begin{aligned} &\frac{2\beta^2 + \beta - 1}{2} \\ &<(2\beta^2 + \beta - 1)\left\{2\alpha - \frac{3}{2} + 4(1-\alpha)\log 2 - 2(1-\alpha)(\log 2)^2\right\} \\ &\leq \gamma \end{aligned}$$

we know that (2.9) contradicts our condition of the theorem. Thus we conclude that $|w(z)| < 1$ for all $z \in U$, that is, that

$$\operatorname{Re}\left\{\frac{zg'(z)}{g(z)}\right\} > \beta \qquad (z \in U). \tag{2.10}$$

This completes the assertion of the theorem.

Letting $\beta = 0$ in Theorem 1, we have

COROLLARY 1. *If* $f(z) \in R(-0.26228\cdots)$, *then* $g(z) \in S^*(0)$, *and if* $f(z) \in R(0)$, *then* $g(z) \in S^*(1/2)$.

REMARK. Corollary 1 is the improvement of Theorem A by Singh and Singh [6]. The first half of Corollary 1 was given by Owa [5].

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