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COMPACT TOEPLITZ OPERATORS WITH BOUNDED SYMBOLS ON THE BERGMAN SPACE

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1. Introduction

The setting of the present paper will be a domain Ω which is a product of balls in the complex n-space $\mathbb{C}^n$. More precisely, Ω is a domain of the form

$$\Omega = \prod_{j=1}^{m} B_{n_j}$$

where each B_{n_j} is the unit ball of $\mathbb{C}^{n_j}$ and $n_1 + \cdots + n_m = n$. We will write V for the volume measure on Ω normalized to have total mass 1. We let $A^2(\Omega)$ denote the Bergman space of square-integrable holomorphic functions on Ω with respect to the measure V. By the mean value property for holomorphic functions it is easy to see that the Bergman space $A^2(\Omega)$ is a closed subspace of $L^2(\Omega) = L^2(\Omega, V)$, so there is an orthogonal projection P — called the Bergman projection — from $L^2(\Omega)$ onto $A^2(\Omega)$. For a function $u \in L^\infty(\Omega)$, the Toeplitz operator T_u with symbol u is defined by

$$T_u f = P(uf)$$

for functions $f \in A^2(\Omega)$. It is clear that the Toeplitz operator T_u is bounded on the Bergman space $A^2(\Omega)$, but not necessarily compact. Answering a question posed by Axler [2], Zheng found a characterization of bounded symbols in terms of a certain vanishing property for corresponding Toeplitz operators to be compact (the original statement in [9] is in a slightly different form):

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THEOREM A (Zheng [9]). *Assume Ω is the ball or the polydisk and let $u \in L^\infty(\Omega)$. Then T_u is compact if and only if*

$$\int_\Omega |P(u \circ \varphi_a)|^p \, dV \to 0 \qquad as \qquad a \to \partial\Omega \qquad for \ all \ (some) \ p \geq 1.$$

Here, the statement $a \to \partial\Omega$ simply means that the euclidean distance $d(a, \partial\Omega)$ between $a \in \Omega$ and the topological boundary $\partial\Omega$ of Ω has the property $d(a, \partial\Omega) \to 0$. Also, for each $a \in \Omega$, φ_a denotes a biholomorphic automorphism of Ω with the property that

$$\varphi_a(a) = 0, \qquad \varphi_a \circ \varphi_a = \text{the identity map.}$$

These notations will have the same meanings for general Ω under consideration (automorphisms φ_a are explicitly described in [6] in the case of the ball, and hence can be defined in an obvious way for general Ω).

In the present paper, we will use a new argument to reprove Zheng's characterization and, at the same time, to give one more characterization. Also, the restriction on the range of p in Zheng's characterization will be removed by a little bit more careful analysis. All these will be done on the setting of a general product of balls. So, in the rest of the paper, Ω will always denote the product of balls mentioned at the beginning of the paper. Our new characterization will be in terms of average vanishing properties over the balls induced by the Bergman metric. Let us first recall the Bergman metric $H_z(a, b)$ defined by

$$H_z(a,b) = \sum_{i,j} \frac{\partial}{\partial z_i} \frac{\partial}{\partial \bar{z}_j} \log K(z,z) a_i \bar{b}_j \qquad (z \in \Omega, \ a, b \in \mathbb{C}^n)$$

where $K(\cdot\,, \cdot)$ denotes the Bergman kernel for Ω (see Section 2). This Bergman metric induces the Bergman distance $\beta(z, w)$ between two points $z, w \in \Omega$ defined by

$$\beta(z,w) = \inf \int_0^1 \sqrt{H_{\gamma(t)}(\gamma'(t), \gamma'(t))} \, dt$$

where the infimum is taken over all C^1-curves γ in Ω such that $\gamma(0) = z$ and $\gamma(1) = w$. This Bergman distance is invariant under automorphisms. Details can be found in [4]. We use the notation $E(a, r)$ for the Bergman metric ball with center at $a \in \Omega$ and radius $r > 0$:

$$E(a,r) = \{z \in \Omega : \beta(a,z) < r\}.$$

We also use the notation $|A|$ for the measure of a Borel subset A of Ω with respect to the measure V. We now state our main result.

THEOREM B. *Let $u \in L^\infty(\Omega)$ and $0 < p < \infty$. Then the following statements are equivalent:*

(*a*) *T_u is compact.*

(*b*) $$\int_\Omega |P(u \circ \varphi_a)|^p \, dV \to 0 \quad \text{as} \quad a \to \partial\Omega.$$

(*c*) $$\frac{1}{|E(a,r)|} \int_{E(a,r)} |P(u \circ \varphi_a)(\varphi_a)|^p \, dV \to 0 \quad \text{as} \quad a \to \partial\Omega$$
for all $r > 0$.

The integrals in condition (*b*) of the above make sense by Lemma 3 of Section 3. The hard part of the proof is the implication (*c*) $\Longrightarrow$ (*a*). To prove it, we will actually estimate the adjoint operator T_u^* by using its integral representation. This idea comes from [7] where Stroethoff used a similar argument to characterize bounded symbols of compact Hankel operators on the Bergman space of the disk.

In Section 2 we collect some basic results about the Bergman kernel and related facts which we use repeatedly throughout this paper. Section 3 is devoted to the proof of Theorem B. Finally in Section 4, we close the paper with some remarks related to Hankel operators and some questions we could not answer.

2. Bergman Kernel

We collect in this section some notations and basic facts which will be used in the sequel. Most of those are well-known and necessary verifications can be found, for example, in [4] or [5].

By the mean value property of holomorphic functions, it is easy to see that point evaluations are bounded linear functionals on the Bergman space $A^2(\Omega)$. Hence there corresponds to every $z \in \Omega$ a unique function $K(\cdot\,, z)$ in $A^2(\Omega)$ — called the Bergman kernel — which has following reproducing property:

$$f(z) = < f, K(\cdot\,, z) > \qquad \text{for all} \qquad f \in A^2(\Omega) \tag{1}$$

where the notation $< , >$ denotes the inner product in $L^2(\Omega)$ with respect to the measure V. Using the well-known formula for the Bergman kernel on the ball, one can easily write down the explicit formula for the Bergman kernel on Ω:

$$K(z,w) = \prod_{j=1}^{m} \frac{1}{(1 - z^j \cdot \bar{w}^j)^{n_j+1}} \qquad (w \in \Omega). \tag{2}$$

Here, we use the notation $z = (z^1, \cdots, z^m)$ with each $z^j = (z_1^j, \cdots, z_{n_j}^j) \in B_{n_j}$ for a point $z \in \Omega$ and

$$z^j \cdot \bar{w}^j = z_1^j \bar{w}_1^j + \cdots + z_{n_j}^j \bar{w}_{n_j}^j$$

for the Hermitian inner product of z^j, $w^j \in \mathbb{C}^{n_j}$. It is often very convenient to use kernels normalized to have L^2-norm 1. So, we let

$$k_a(z) = \frac{K(z,a)}{\sqrt{K(a,a)}} \qquad (a, z \in \Omega).$$

There is a well-known transformation formula for the Bergman kernels on biholomorphically equivalent domains. In particular, for the automorphisms φ_a of Ω, we have

$$K(\varphi_a(z), \varphi_a(w))(J\varphi_a)(z)\overline{(J\varphi_a)(w)} = K(z,w) \qquad (z, w \in \Omega) \tag{3}$$

where $J\varphi_a$ denotes the complex Jacobian determinant of φ_a. The special case $z = w$ yields a useful identity

$$K(\varphi_a(z), \varphi_a(z))|(J\varphi_a)(z)|^2 = K(z,z) \qquad (z \in \Omega). \tag{4}$$

The real Jacobian determinant of φ_a turns out to be the same as $|J\varphi_a|^2$ for which we have the identity

$$|J\varphi_a|^2 = |k_a|^2 \tag{5}$$

on Ω. This follows from a straightforward calculation by using transformation formulas (3), (4) and the fact $K(\cdot\,, 0) = 1$. Since φ_a is an involution, another straightforward calculation shows

$$k_a(\varphi_a(z))k_a(z) = 1 \qquad (z \in \Omega). \tag{6}$$

A manipulation of the identities (4) and (6) shows

$$\begin{aligned}|K(z,\varphi_z(w))|^\alpha |k_z(w)|^2 K(\varphi_z(w),\varphi_z(w))^\mu \\ = |K(z,w)|^{2-\alpha-2\mu} K(z,z)^{\alpha+\mu-1} K(w,w)^\mu\end{aligned}$$

and thus, by (5), we have a change-of-variable formula:

$$\begin{aligned}&\int_\Omega h(w)|K(z,w)|^\alpha K(w,w)^\mu \, dV(w) \\ &\quad = K(z,z)^{\alpha+\mu-1} \int_\Omega h(\varphi_z(w))|K(z,w)|^{2-\alpha-2\mu} K(w,w)^\mu \, dV(w)\end{aligned} \tag{7}$$

for all α, μ real whenever the integrals make sense.

3. Compact Toeplitz Operators

We start with an observation on how Toeplitz operators act on the Bergman kernel.

PROPOSITION 1. *Let $u \in L^\infty(\Omega)$. Then we have*

$$T_u k_a = [P(u \circ \varphi_a) \circ \varphi_a] k_a$$

for all $a \in \Omega$.

Proof. Fix a point $a \in \Omega$. Define a linear operator U_a on $L^2(\Omega)$ by

$$U_a f = (f \circ \varphi_a) k_a$$

for $f \in L^2(\Omega)$. Using (5) and (6), one can readily see that U_a is a unitary operator taking $A^2(\Omega)$ onto itself. It follows that

$$U_a P = P U_a \tag{8}$$

on $L^2(\Omega)$. Since φ_a is an involution, we obtain from (8) that

$$\begin{aligned}T_u k_a &= P(u k_a) \\ &= P U_a (u \circ \varphi_a) \\ &= U_a P (u \circ \varphi_a) \\ &= [P(u \circ \varphi_a) \circ \varphi_a] k_a.\end{aligned}$$

This completes the proof. □

As is well-known, a bounded operator on a Hilbert space is compact if and only if its adjoint operator is compact. So, when we prove the compactness of T_u, we will actually show that its adjoint operator $T_u^* : A^2(\Omega) \to A^2(\Omega)$ is compact. The following proposition gives a convenient way to represent the operator T_u^*.

PROPOSITION 2. *Let $u \in L^\infty(\Omega)$. Then we have*

$$(T_u^* h)(a) = \int_\Omega h(w)\overline{P(u \circ \varphi_a)(\varphi_a(w))}K(a,w)\,dV(w)$$

for $h \in A^2(\Omega)$ and $a \in \Omega$.

Proof. Let $h \in A^2(\Omega)$ and $a \in \Omega$. Then, by the reproducing property (1) of the Bergman kernel, we have

$$(T_u^* h)(a) = < T_u^* h, K(\cdot\,, a) > = < h, T_u K(\cdot\,, a) > .$$

Note that $\overline{K(\cdot\,, a)} = K(a, \cdot)$. The integral representation thus follows from Proposition 1. □

Note. We remark in passing that Propositions 1 and 2 remain valid on arbitrary bounded symmetric domains by the same proof.

Before turning to the proof of our main result Theorem B, we need several lemmas. First, we show that the Bergman projection P is a bounded operator from $L^\infty(\Omega)$ into $L^p(\Omega)$.

LEMMA 3. *For any $0 < p < \infty$, the operator $P : L^\infty(\Omega) \to L^p(\Omega)$ is bounded.*

Proof. Let $h \in L^\infty(\Omega)$ and $z \in \Omega$. By the reproducing property (1) we have

$$Ph(z) = < Ph, K(\cdot\,, z) > = < h, K(\cdot\,, z) >$$

and thus, by (2) and Fubini's theorem, we obtain

$$|Ph(z)| \le \|h\|_\infty \prod_{j=1}^m \int_{B_{n_j}} \frac{1}{|1 - z^j \cdot \bar{w}^j|^{n_j+1}}\, dV_j(w^j)$$

where each V_j denotes the normalized volume measure on B_{n_j}. Now Proposition 1.4.10 of [6] yields

$$|Ph(z)| \le C||h||_\infty \prod_{j=1}^{m} \left(1 + \log \frac{1}{1-|z^j|^2}\right) \qquad (z \in \Omega)$$

for some constant C depending only on Ω. Another application of Fubini's theorem therefore shows

$$\int_\Omega |Ph|^p \, dV \le C^p ||h||_\infty^p \prod_{j=1}^{m} \int_{B_{n_j}} \left(1 + \log \frac{1}{1-|z^j|^2}\right)^p dV_j(z^j)$$

for every $0 < p < \infty$. Note that the integrals on the right side of the above are finite. The proof is complete. □

The following is a version of Proposition 1.4.10 in [6] for the unit ball.

LEMMA 4. *There are constants $\mu > 0$ and $q > 1$ such that*

$$\sup \int_\Omega |K(a,z)|^{q(1-2\mu)} K(z,z)^{q\mu} \, dV(z) < \infty$$

where the supremum is taken over all $a \in \Omega$.

Proof. See Lemma 9 of [3].

LEMMA 5. *Let $h \in L^\infty(\Omega)$ and $\mu > 0$ be as in Lemma 4. Then there exists a constant C, depending only on Ω, such that*

$$\int_\Omega |Ph(\varphi_z(w))|^2 |K(z,w)| K(w,w)^\mu \, dV(w) \le C||h||_\infty^2 K(z,z)^\mu$$

for all $z \in \Omega$.

Proof. Let $q > 1$ be as in Lemma 4 and $1/p + 1/q = 1$. Apply the change-of-variable formula (7) and then Hölder's inequality to obtain

$$\int_\Omega |Ph(\varphi_z(w))|^2 |K(z,w)| K(w,w)^\mu \, dV(w)$$

$$= K(z,z)^\mu \int_\Omega |Ph(w)|^2 |K(z,w)|^{1-2\mu} K(w,w)^\mu \, dV(w)$$

$$\leq K(z,z)^{\mu}\left(\int_{\Omega}|Ph|^{2p}\,dV\right)^{1/p}$$

$$\times\left(\int_{\Omega}|K(z,w)|^{q(1-2\mu)}K(w,w)^{q\mu}\,dV(w)\right)^{1/q}$$

for all $z\in\Omega$. Thus the result follows from Lemma 4 and the inequality

$$\int_{\Omega}|Ph|^{2p}\,dV\leq C\|h\|_{\infty}^{2p}$$

for some constant C depending only on Ω, which is a consequence of Lemma 3. The proof is complete. □

We will need some informations about the size of the volume of the Bergman metric balls $E(a,r)$.

LEMMA 6. *For $r>0$, there are positive constants $C(r)$, $c(r)$ so that*

$$c(r)\leq\frac{|k_a(w)|^2}{|E(a,r)|}\leq C(r)$$

for all $a\in\Omega$ and $w\in E(0,r)$.

Proof. Fix $r>0$. Since $K(\cdot\,,\,\cdot)$ is continuous and nonvanishing on the compact set $\bar{E}(0,r)\times\bar{\Omega}$, we have

$$m_r=\inf|K(w,a)|^2>0,\qquad M_r=\sup|K(w,a)|^2<\infty$$

where the infimum and supremum are taken over all $w\in E(0,r)$, $a\in\Omega$. Now, since φ_a is an involution and the Bergman distance is automorphism-invariant, one can easily see that $\varphi_a E(a,r)=E(0,r)$. Since $|J\varphi_a|^2=|k_a|^2$ is the real Jacobian determinant of φ_a, a change of variables shows that

$$|E(a,r)|=\int_{E(a,r)}dV=\int_{E(0,r)}|k_a|^2\,dV$$

and therefore

$$\begin{aligned}|E(a,r)| &\geq m_r|E(0,r)|K(a,a)^{-1} \\ &= m_r|E(0,r)||K(w,a)|^{-2}|k_a(w)|^2 \\ &\geq m_r M_r^{-1}|E(0,r)||k_a(w)|^2\end{aligned}$$

for all $a \in \Omega$ and $w \in E(0,r)$. Similarly, we have

$$|E(a,r)| \leq m_r^{-1} M_r |E(0,r)||k_a(w)|^2$$

for all $a \in \Omega$ and $w \in E(0,r)$. The proof is complete. □

LEMMA 7. *For $r > 0$, there are positive constants $C(r)$, $c(r)$ so that*

$$c(r) \leq |k_a(w)|^2|E(a,r)| \leq C(r)$$

for all $a \in \Omega$ and $w \in E(a,r)$.

Proof. Since $\varphi_a E(a,r) = E(0,r)$, we have $\varphi_a(w) \in E(0,r)$ for $w \in E(a,r)$. Thus, the lemma follows from Lemma 6 and (6). □

We are now ready to prove a preliminary version of our main result.

THEOREM 8. *Let $u \in L^\infty(\Omega)$. Then the following statements are equivalent:*

(a) *T_u is compact.*

(b) $$\int_\Omega |P(u \circ \varphi_a)|^2 \, dV \to 0 \quad \text{as} \quad a \to \partial\Omega.$$

(c) $$\frac{1}{|E(a,r)|} \int_{E(a,r)} |P(u \circ \varphi_a)(\varphi_a)|^2 \, dV \to 0 \quad \text{as} \quad a \to \partial\Omega$$
for all $r > 0$.

In the proof the same letter C stands for various constants which may change with each occurrence.

Proof. (a) $\Longrightarrow$ (b) : Suppose that T_u is compact on $A^2(\Omega)$. Since the normalized kernel k_a converges uniformly to 0 on compact subsets

of Ω as $a \to \partial\Omega$, one can easily see that k_a converges weakly to 0 in $A^2(\Omega)$ as $a \to \partial\Omega$. The compactness of T_u therefore implies

$$\int_\Omega |T_u k_a|^2 \, dV \to 0 \qquad \text{as} \qquad a \to \partial\Omega. \tag{9}$$

On the other hand, using Proposition 1, we can easily see by a change of variables that

$$\begin{aligned} \int_\Omega |P(u \circ \varphi_a)|^2 \, dV &= \int_\Omega |P(u \circ \varphi_a)(\varphi_a)|^2 |k_a|^2 \, dV \\ &= \int_\Omega |T_u k_a|^2 \, dV. \end{aligned}$$

Combining the above with (9), we have (b).

$(b) \implies (c)$: Suppose that (b) holds, and let $r > 0$. Recall that $\varphi_a E(a,r) = E(0,r)$ for $a \in \Omega$. Using a change of variables and Lemma 6, one obtains

$$\begin{aligned} &\frac{1}{|E(a,r)|} \int_{E(a,r)} |P(u \circ \varphi_a)(\varphi_a)|^2 \, dV \\ &\qquad = \frac{1}{|E(a,r)|} \int_{E(0,r)} |P(u \circ \varphi_a)|^2 |k_a|^2 \, dV \\ &\qquad \le C \int_\Omega |P(u \circ \varphi_a)|^2 \, dV \end{aligned}$$

for some constant C independent of a, so that (c) follows.

$(c) \implies (a)$: We will assume (c) and construct a sequence of compact operators which converges in the opertaor norm to the operator T_u^*. Then the compactness of T_u will follow from that of T_u^*.

For $\rho > 0$, put

$$\Omega_\rho = \{z \in \Omega : d(z, \partial\Omega) \ge \rho\}.$$

Recall that $d(\cdot\ ,\ \cdot)$ denotes the euclidean distance. Let M_ρ be the multiplication by the characteristic function of Ω_ρ, acting on $L^2(\Omega)$.

Since the symbol of M_ρ is supported on the compact set Ω_ρ, the opetator M_ρ is compact when restricted to $A^2(\Omega)$. Thus the operator $M_\rho T_u^* : A^2(\Omega) \to L^2(\Omega)$ is compact. Put

$$G_\rho = T_u^* - M_\rho T_u^*$$

for simplicity. Now we show that the operator norm $||G_\rho||$ converges to 0 as $\rho \to 0$. To do so, pick any $h \in A^2(\Omega)$. Then, by Proposition 2, we have

$$G_\rho h(a) = \chi_\rho(a) \int_\Omega h(w)\overline{P(u \circ \varphi_a)(\varphi_a(w))}K(a,w)\, dV(w)$$

where χ_ρ denotes the characteristic function of the set $\Omega \setminus \Omega_\rho$. Given $r > 0$, decompose

$$G_\rho = U_{\rho,r} + V_{\rho,r} \tag{10}$$

where

$$U_{\rho,r}h(a) = \chi_\rho(a) \int_{E(a,r)} h(w)\overline{P(u \circ \varphi_a)(\varphi_a(w))}K(a,w)\, dV(w)$$

and

$$V_{\rho,r}h(a) = \chi_\rho(a) \int_{\Omega \setminus E(a,r)} h(w)\overline{P(u \circ \varphi_a)(\varphi_a(w))}K(a,w)\, dV(w).$$

We first estimate the operator $U_{\rho,r}$. Put

$$I(a,r) = \frac{1}{|E(a,r)|} \int_{E(a,r)} |P(u \circ \varphi_a)(\varphi_a)|^2\, dV \qquad (a \in \Omega,\ r > 0)$$

for simplicity. By the Cauchy-Schwarz inequality

$$|U_{\rho,r}h(a)|^2 \le \chi_\rho(a)\left(\int_{E(a,r)} |P(u \circ \varphi_a)(\varphi_a(w))h(w)K(a,w)|\, dV(w)\right)^2$$

$$\le \chi_\rho(a) I(a,r) \int_{E(a,r)} |h(w)|^2 |E(a,r)||K(a,w)|^2\, dV(w).$$

An application of Fubini's theorem therefore yields

$$\int_\Omega |U_{\rho,r}h|^2\,dV \le \Big(\sup_{a\in\Omega\setminus\Omega_\rho} I(a,r)\Big) \times \int_\Omega |h(w)|^2 \int_{E(w,r)} |E(a,r)||K(a,w)|^2\,dV(a)\,dV(w).$$

By Lemma 7 and (7), one can estimate the inner integral of the right side of the above:

$$\begin{aligned}\int_{E(w,r)} |E(a,r)||K(a,w)|^2\,dV(a) &\le C\int_{E(w,r)} K(a,a)\,dV(a)\\ &= C\int_{E(0,r)} K(z,z)\,dV(z)\\ &\le C\end{aligned}$$

for some constants C independent of ρ. Here, the last inequality holds because the function $z \to K(z,z)$ is continuous and hence bounded on the compact set $\bar{E}(0,r)$. Therefore we get the following estimate for the operator norm of $U_{\rho,r}$:

$$||U_{\rho,r}||^2 \le C\Big(\sup_{a\in\Omega\setminus\Omega_\rho} I(a,r)\Big)$$

for each $r > 0$ and for some constant C independent of ρ. It follows from the assumption that

$$||U_{\rho,r}|| \to 0 \qquad \text{as} \qquad \rho \to 0 \tag{11}$$

for each r.

Now we estimate the operator $V_{\rho,r}$. Let $\mu > 0$ be the constant as in Lemma 4. Then, by the Cauchy-Schwarz inequality again,

$$|V_{\rho,r}h(a)|^2 \le \left(\int_{\Omega\setminus E(a,r)} |P(u\circ\varphi_a)(\varphi_a(w))h(w)K(a,w)|\,dV(w)\right)^2$$

$$\leq \left(\int_{\Omega \setminus E(a,r)} |P(u \circ \varphi_a)(\varphi_a(w))|^2 |K(a,w)| K(w,w)^{\mu} \, dV(w) \right)$$

$$\times \left(\int_{\Omega \setminus E(a,r)} |h(w)|^2 |K(a,w)| K(w,w)^{-\mu} \, dV(w) \right)$$

$$\leq C ||u||_{\infty}^2 K(a,a)^{\mu}$$

$$\times \int_{\Omega \setminus E(a,r)} |h(w)|^2 |K(a,w)| K(w,w)^{-\mu} \, dV(w)$$

where C is a constant, depending only on Ω, provided by Lemma 5. Then, by an application of Fubini's theorem again, we have

$$\int_{\Omega} |V_{\rho,r} h|^2 \, dV \leq C ||u||_{\infty}^2 \int_{\Omega} |h(w)|^2 K(w,w)^{-\mu}$$

$$\times \int_{\Omega \setminus E(w,r)} K(a,a)^{\mu} |K(a,w)| \, dV(a) \, dV(w).$$

Let

$$J(w,r) = K(w,w)^{-\mu} \int_{\Omega \setminus E(w,r)} K(a,a)^{\mu} |K(a,w)| \, dV(a). \tag{12}$$

Then we have

$$\int_{\Omega} |V_{\rho,r} h|^2 \, dV \leq C ||u||_{\infty}^2 \left(\sup_{w \in \Omega} J(w,r) \right) \int_{\Omega} |h|^2 \, dV.$$

In other words,

$$||V_{\rho,r}||^2 \leq C ||u||_{\infty}^2 \left(\sup_{w \in \Omega} J(w,r) \right) \tag{13}$$

for some constant C depending only on Ω. Now, make subsitution $a = \varphi_w(z)$ in the integral of the right side of (12) and use (7), to see that $J(w,r)$ is exactly the same as the integral

$$\int_{\Omega \setminus E(0,r)} K(z,z)^{\mu} |K(w,z)|^{1-2\mu} \, dV(z).$$

Choose $q > 1$ such that $1/p + 1/q = 1$ where $q > 1$ is the constant as in Lemma 4. It then follows from Hölder's inequality and Lemma 4 that

$$\sup_{w\in\Omega} J(w,r) \le C|\Omega \setminus E(0,r)|^{1/p}$$

for some constant C depending only on Ω. Since $\Omega = \bigcup_{r>0} E(0,r)$, it follows from the above that

$$\sup_{w\in\Omega} J(w,r) \to 0 \qquad \text{as} \qquad r \to \infty$$

and therefore, by (13),

$$\sup_{\rho>0} ||V_{\rho,r}|| \to 0 \qquad \text{as} \qquad r \to \infty.$$

This, together with (10) and (11), gives

$$||G_\rho|| \to 0 \qquad \text{as} \qquad \rho \to 0$$

as desired. The proof is complete. □

Before proving Theorem B, we have a couple of simple lemmas.

LEMMA 9. *Let $u \in L^\infty(\Omega)$. Then the following statements are equivalent:*

(a) $\displaystyle\int_\Omega |P(u\circ\varphi_a)|^2\, dV \to 0 \quad \text{as} \quad a \to \partial\Omega.$

(b) $\displaystyle\int_\Omega |P(u\circ\varphi_a)|^p\, dV \to 0 \quad \text{as} \quad a \to \partial\Omega \quad \text{for all} \quad 0 < p < \infty.$

(c) $\displaystyle\int_\Omega |P(u\circ\varphi_a)|^p\, dV \to 0 \quad \text{as} \quad a \to \partial\Omega \quad \text{for some} \quad 0 < p < \infty.$

Proof. First assume (a) and show (b). We may further assume $p > 2$ by Jensen's inequality. Then, by the Cauchy-Schwarz inequality we have

$$\left(\int_\Omega |P(u\circ\varphi_a)|^p\, dV\right)^2 \le \left(\int_\Omega |P(u\circ\varphi_a)|^2\, dV\right)\left(\int_\Omega |P(u\circ\varphi_a)|^{2p-2}\, dV\right).$$

By Lemma 3 the second integral of the right side of the above stays bounded independently of a, and therefore (b) follows.

The implication $(b) \implies (c)$ is trivial. Finally assume (c) and show (a). By Jensen's inequality once more, we only need consider the case $p < 2$. For such p, another application of the Cauchy-Schwarz inequality shows

$$\left(\int_\Omega |P(u\circ\varphi_a)|^2\,dV\right)^2 \le \left(\int_\Omega |P(u\circ\varphi_a)|^p\,dV\right)\left(\int_\Omega |P(u\circ\varphi_a)|^{4-p}\,dV\right),$$

and thus (a) holds by Lemma 3 as before. □

LEMMA 10. *Let $u \in L^\infty(\Omega)$ and $r > 0$. Then the following statements are equivalent:*

$$(a) \quad \frac{1}{|E(a,r)|}\int_{E(a,r)} |P(u\circ\varphi_a)(\varphi_a)|^2\,dV \to 0 \quad \text{as} \quad a \to \partial\Omega.$$

$$(b) \quad \frac{1}{|E(a,r)|}\int_{E(a,r)} |P(u\circ\varphi_a)(\varphi_a)|^p\,dV \to 0 \quad \text{as} \quad a \to \partial\Omega$$
for all $0 < p < \infty$.

$$(c) \quad \frac{1}{|E(a,r)|}\int_{E(a,r)} |P(u\circ\varphi_a)(\varphi_a)|^p\,dV \to 0 \quad \text{as} \quad a \to \partial\Omega$$
for some $0 < p < \infty$.

Proof. Assume (a) and show (b). We only consider the case $p > 2$ by Jensen's inequality. By the Cauchy-Schwarz inequality we have

$$\left(\frac{1}{|E(a,r)|}\int_{E(a,r)} |P(u\circ\varphi_a)(\varphi_a)|^p\,dV\right)^2$$

$$\le \left(\frac{1}{|E(a,r)|}\int_{E(a,r)} |P(u\circ\varphi_a)(\varphi_a)|^2\,dV\right)$$

$$\times \left(\frac{1}{|E(a,r)|}\int_{E(a,r)} |P(u\circ\varphi_a)(\varphi_a)|^{2p-2}\,dV\right).$$

The second integral of the right side of the above stays bounded independently of a by a change of variables and Lemmas 3 and 6 as follows.

$$\begin{aligned}
&\frac{1}{|E(a,r)|}\int_{E(a,r)} |P(u\circ\varphi_a)(\varphi_a)|^{2p-2}\,dV \\
&\qquad = \frac{1}{|E(a,r)|}\int_{E(0,r)} |P(u\circ\varphi_a)|^{2p-2}|k_a|^2\,dV \\
&\qquad \le C\int_{\Omega} |P(u\circ\varphi_a)|^{2p-2}\,dV \\
&\qquad \le C\|u\|_{\infty}^{2p-2}
\end{aligned}$$

for some constant C independent of a. Thus (b) holds.

The implication $(b) \implies (c)$ is trivial. Now assume (c) and show (a). By Jensen's inequality again, we may further assume $p < 2$. Then, by the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
&\left(\frac{1}{|E(a,r)|}\int_{E(a,r)} |P(u\circ\varphi_a)(\varphi_a)|^2\,dV\right)^2 \\
&\le \left(\frac{1}{|E(a,r)|}\int_{E(a,r)} |P(u\circ\varphi_a)(\varphi_a)|^p\,dV\right) \\
&\qquad\qquad \times \left(\frac{1}{|E(a,r)|}\int_{E(a,r)} |P(u\circ\varphi_a)(\varphi_a)|^{4-p}\,dV\right) \\
&\le C\frac{1}{|E(a,r)|}\int_{E(a,r)} |P(u\circ\varphi_a)(\varphi_a)|^p\,dV
\end{aligned}$$

for some constant C independent of a as before. Thus (a) follows as desired. □

Now, our main theorem is just a simple consequence of Theorem 8, Lemma 9, and Lemma 10.

Proof of Theorem B. Combining Theorem 8 with Lemma 9, we have the equivalence of (a) and (b). On the other hand, the equivalence of

conditions (a) and (c) follows from Theorem 8 and Lemma 10. The proof is complete. □

4. Remarks

Let $Q = I - P$ be the orthogonal projection of $L^2(\Omega)$ onto $A^2(\Omega)^\perp$, the orthogonal complement of $A^2(\Omega)$ in $L^2(\Omega)$. For $u \in L^\infty(\Omega)$, the Hankel operator H_u with symbol u is defined by

$$H_u f = Q(uf)$$

for $f \in A^2(\Omega)$. Clearly H_u is a bounded linear operator of $A^2(\Omega)$ into $A^2(\Omega)^\perp$. Answering a question posed by Axler [1], Stroethoff first characterized bounded symbols of compact Hankel operators. In [7] Stroethoff proved the disk version of the following theorem for $1 < p < \infty$. Later Stroethoff [8] used a similar method on the ball and the polydisk to obtain the equivalence of conditions (a) and (b) for $p = 2$ of the following theorem and pointed out that the result remains valid on general products of balls. Zheng [9] independently obtained a result similar to Theorem A. However, repeating the argument of the present paper with H_u and Q in place of T_u and P, respectively, one can prove the following theorem without any restricion on the range of p:

THEOREM 11. *Let $u \in L^\infty(\Omega)$ and $0 < p < \infty$. Then the following statements are equivalent:*

(a) *H_u is compact.*

(b) $$\int_\Omega |Q(u \circ \varphi_a)|^p \, dV \to 0 \quad \text{as} \quad a \to \partial\Omega.$$

(c) $$\frac{1}{|E(a,r)|} \int_{E(a,r)} |Q(u \circ \varphi_a)(\varphi_a)|^p \, dV \to 0 \quad \text{as} \quad a \to \partial\Omega$$
for all $r > 0$.

In view of definitions of Toeplitz and Hankel operators, it is interesting that characterizations in Theorem B and Theorem 11 are completely parallel. Let us observe a simple consequence. There are well-known characterizations of positive symbols for corresponding (densely-defined) Toeplitz operators to be compact. One of them

is the boundary vanishing property of Berezin transforms of symbols. More precisely, for $v \geq 0$, $v \in L^1(\Omega)$, the following two conditions are known to be equivalent (see [10] for bounded symmetric domains):

- T_v is compact.
- $\int_\Omega (v \circ \varphi_a)\, dV \to 0 \quad \text{as} \quad a \to \partial\Omega.$

For $u \in L^\infty(\Omega)$, apply this characterization to $v = |u|^2$ and use the fact

$$\int_\Omega |(u \circ \varphi_a)|^2 dV = \int_\Omega |P(u \circ \varphi_a)|^2 dV + \int_\Omega |Q(u \circ \varphi_a)|^2 dV$$

to see from Theorem B and Theorem 11 (with $p = 2$) that $T_{|u|^2}$ is compact if and only if T_u and H_u are both compact. On the other hand, since $u \in L^\infty(\Omega)$, $T_{|u|^2}$ is compact if and only if $T_{|u|}$ is compact. Thus we have

COROLLARY 12. *Let $u \in L^\infty(\Omega)$. Then the following statements are equivalent:*

(a) *$T_{|u|}$ is compact.*

(b) *T_u and H_u are both compact.*

(c) $\int_\Omega |u \circ \varphi_a|\, dV \to 0 \quad \text{as} \quad a \to \partial\Omega.$

(d) $\dfrac{1}{|E(a,r)|} \displaystyle\int_{E(a,r)} |u|\, dV \to 0 \quad \text{as} \quad a \to \partial\Omega$ *for all (some) $r > 0$.*

The equivalence of conditions (a) and (d) of the above, as well as some other equivalent conditions, can also be found in [10].

We now close the paper with a couple of questions. In our proof of Theorem B, the boundedness of symbols play a crucial role. We do not know whether such boundedness hypothesis is essential. For example, does Theorem 8 hold with $u \in L^2(\Omega)$ in place of $u \in L^\infty(\Omega)$? For $u \in L^2(\Omega)$, we do not even know whether the condition

$$\sup_{a \in \Omega} \int_\Omega |P(u \circ \varphi_a)|^2 dV < \infty$$

is necessary and sufficient for T_u to be bounded. However, it must be pointed out that whatever condition on u that is necessary and sufficient for T_u to be bounded must be automorphism-invariant, since T_u and $T_{u\circ\varphi_a}$ are unitarily equivalent:

$$U_a T_u U_a = T_{u\circ\varphi_a}$$

where U_a denotes the unitary operator on $A^2(\Omega)$ defined in the proof of Proposition 1.

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