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THE GLOBALLY REGULAR SOLUTIONS OF SEMILINEAR WAVE EQUATIONS WITH A CRITICAL NONLINEARITY

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0. Introduction

In this paper we study the existence of a globally regular solution of the semilinear wave equation with a critical nonlinearity

$$u_{tt} - \Delta u + u^3 = 0, \tag{0.1}$$

where $u(x,t) : R^4 \times R \to R$ is a function of four space variables and time. In order to solve (0.1) one has to prescribe initial data at a fixed time $t = 0$, i.e.

$$u(x,0) = u_0(x), u_t(x,0) = u_1(x). \tag{0.2}$$

The equation (0.1) is a special case of a more general set of model equations

$$u_{tt} - \Delta u + |u|^{p-1}u = 0, \tag{0.3}$$

where $u(x,t) : R^n \times R \to R$ is a function.

In case $n = 3$ and $p < 5$, Jörgens[3] proved in 1961 that the nonlinear wave equation (0.3) with initial data

$$u(x,0) = u_0(x) \in C^3(R^3),\ u_t(x,0) = u_1(x) \in C^2(R^3) \tag{0.4}$$

has a globally unique C^2 solution. In case $n = 3$ and $p = 5$(critical power), Rauch[4] in 1981 first proved the existence of a global C^2 solution provided the initial energy is small enough. In 1988 Struwe [5][6] proved the existence of a radially symmetric global C^2 solution

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provided the initial data is radially symmetric. Finally, Grillakis[2] in 1990 was able to remove symmetric assumption in Struwe's result. In case $n < 13$, the equation (0.3) with suitable initial data has a global C^2 solution provided $p < \frac{n+2}{n-2}$ (See [1]).

In this paper we shall prove

THEOREM 0.1. *Let $u_0 \in C^4(R^4), u_1 \in C^3(R^4)$ be arbitrary initial data. If $u \in C^2(R^4 \times [0,T))$, for some $T > 0$, is a solution of (0.1) and (0.2), then there exists a solution $u \in C^2(R^4 \times [0,\infty))$ to the Cauchy problem (0.1) and (0.2).*

The proof is divided into several parts. In Section 1, we shall establish an integral representation of the solution of a semilinear wave equation. In Section 2, using the Hardy type inequality we prove the existence of a global C^2 solution with small initial data. In Section 3, we apply the identities to derive the several estimates of solutions. In Section 4, we shall prove the existence of a global C^2 solution with arbitrary initial data.

We shall use the following notations: Let $z = (x,t)$ denote a point in the space -time $R^4 \times R$. Given $z_0 = (x_0, t_0)$, let

$$K(z_0) = \{z = (x,t) : \ |x - x_0| \leq t_0 - t\}$$

be the forward(backward) light cone with vertex at z_0,

$$M(z_0) = \{z = (x,t) : \ |x - x_0| = t_0 - t\}$$

its mantle, and

$$D(t, z_0) = \{z = (x,t) \in K(z_0)\} \quad (t \text{ fixed})$$

its time-like sections. If $z_0 = (0,0)$, z_0 will be omitted. For any space-time region $Q \subset R^4 \times R$ and $T < S$, we let

$$Q_T^S = \{z = (x,t) \in Q : \ T \leq t \leq S\}$$

the truncated region. Hence, for instance, we have

$$\partial K_t^s = D(s) \cup D(t) \cup M_t^s.$$

If $s = \infty$ or $t = -\infty$, it will be omitted. For $x_0 \in R^4$, let

$$B_R(x_0) = \{x \in R^4 : \ |x - x_0| \leq R\}$$

with boundary

$$S_R(x_0) = \{x \in R^4 : \ |x - x_0| = R\}.$$

1. Integral Representation

In this section we shall give an integral representation of a solution of the semilinear wave equation

$$u_{tt} - \Delta u + u^3 = 0 \qquad \text{in } R^4 \times R \tag{1.1}$$

with initial data

$$u(x,0) = u_0(x), \; u_t(x,0) = u_1(x). \tag{1.2}$$

Assume that u is a solution belonging to $C^2(R^4 \times [0,T))$ of (1.1) and (1.2). Let x_0 and x be points in R^4. Let $y = x - x_0$ where x_0 is a fixed point and x is a variable. Define the functions $[u]$ as

$$[u] = u(x, t - |y|).$$

Then

$$\nabla[u] = [\nabla u] - [u_t],$$

$$\Delta[u] = [\Delta u] - 2[\nabla u_t] \cdot \hat{y} + [u_{tt}] - \frac{3}{|y|}[u_t],$$

$$\nabla[u_t] = [\nabla u_t] - [u_{tt}] \cdot \hat{y},$$

where $\hat{y} = \frac{y}{|y|}$ is the unit vector of y. Eliminating $[\nabla u_t]$ from the above, we have

$$\Delta[u] + 2\hat{y} \cdot \nabla[u_t] + \frac{3}{|y|}[u_t] = [\Delta u] - [u_{tt}] = [u^3]. \tag{1.3}$$

Multiply (1.3) by $\frac{1}{|y|^2}$ to get the identity

$$\nabla \cdot \left\{ \frac{1}{|y|^2}[\nabla u] + \frac{y}{|y|^3}[u_t] + \frac{2}{|y|^4} y[u] \right\} + \frac{1}{|y|^3}[u_t] = \frac{1}{|y|^2}[u^3]. \tag{1.4}$$

Take $z_0 = (x_0, t_0)$ such that $|x_0| \le t_0$ and $t_0 < T$ and integrate (1.4) inside the domain Λ bounded by the surfaces $S_\epsilon = \{|y| = \epsilon\}, S = \{|y| = t_0\}$. Then

$$\int_\Lambda \nabla \cdot \left\{ \frac{1}{|y|^2}[\nabla u] + \frac{y}{|y|^3}[u_t] + \frac{2y}{|y|^4}[u] \right\} dy = \int_\Lambda \left\{ -\frac{1}{|y|^3}[u_t] + \frac{1}{|y|^2}[u^3] \right\} dy.$$

The divergence theorem gives

$$\int_{|y|=t_0} \frac{1}{|y|^2}\Big\{\hat{y}\cdot\nabla u(x,0) + u_t(x,0) + \frac{2}{|y|}u(x,0)\Big\}\, do$$
$$-\int_{|y|=\epsilon} \frac{1}{|y|^2}\Big\{\hat{y}\cdot\nabla u(x,t_0-\epsilon) + u_t(x,t_0-\epsilon) + \frac{2}{|y|}u(x,t_0-\epsilon)\Big\}\, do$$
$$=\int_{\epsilon<|y|<t_0}\Big\{-\frac{1}{|y|^3}u_t(x,t_0-|y|) + \frac{1}{|y|^2}u^3(x,t_0-|y|)\Big\}\, dy.$$

By letting $\epsilon \to 0$ we have

$$\int_{|y|=t_0} \frac{1}{|y|^2}\Big\{\nabla u(x,0)\cdot\hat{y} + u_t(x,0) + \frac{2}{|y|}u(x,0)\Big\}\, do - 4\omega_4 u(x_0,t_0)$$
$$=\int_{|y|<t_0}\Big\{-\frac{1}{|y|^3}u_t + \frac{1}{|y|^2}u^3\Big\}\, dy. \tag{1.5}$$

Thus we have

$$\begin{aligned} u(x_0,t_0) =& \frac{1}{2\omega_4}\int_{|y|=t_0}\frac{1}{|y|^2}\Big\{\nabla u_0\cdot\hat{y} + u_1 + \frac{2}{|y|}u_0\Big\}\, do \\ &+ \frac{1}{2\omega_4}\int_{|y|<t_0}\frac{1}{|y|^3}u_t(x,t_0-|y|)\, dy \\ &- \frac{1}{2\omega_4}\int_{|y|<t_0}\frac{1}{|y|^2}u^3(x,t_0-|y|)\, dy \\ =& u_L(x_0,t_0) + u_N(x_0,t_0), \end{aligned}$$

where the linear part of $u(x_0,t_0)$ is given by

$$\begin{aligned} u_L(x_0,t_0) =& \frac{1}{2\omega_4}\int_{|y|=t_0}\frac{1}{|y|^2}\Big\{\nabla u_0\cdot\hat{y} + u_1 + \frac{2}{|y|}u_0\Big\}\, do \\ &+ \frac{1}{2\omega_4}\int_{|y|<t_0}\frac{u_t(x,t_0-|y|)}{|y|^3}\, dy \end{aligned} \tag{1.7}$$

and the nonlinear part of $u(x_0,t_0)$ is given by

$$u_N(x_0,t_0) = -\frac{1}{2\omega_4}\int_{|y|<t_0} \frac{u^3(x,t_0-|y|)}{|y|^2}\,dy. \tag{1.8}$$

Let $z_0 = (x_0, t_0)$ and $z = (x,t)$ for $z \in M_0^{t_0}(z_0) = \{(x,t) : |x - x_0| = t_0 - t,\ 0 \le t \le t_0\}$. Then $z - z_0 = (y, |y|)$ and

$$u_N(x_0,t_0) = -\frac{1}{\sqrt{2}\omega_4}\int_{M_0^{t_0}(z_0)} \frac{u^3(z)}{|z-z_0|^2}\,do \tag{1.9}$$

Thus we have proved the

THEOREM 1.1. *Let $u \in C^2(R^4 \times [0,T))$ be a solution of (1.1) and (1.2). Then for every $z_0 \in K_0^T = \{(x,t)||x| \le T - t, 0 < t \le T\}$, u satisfies the integral equation*

$$u(z_0) = u_L(z_0) + u_N(z_0), \tag{1.10}$$

where $u_L(z_0)$ and $u_N(z_0)$ are given by (1.7) and (1.9).

2. Globally Regular Solutions for the Small Initial Data

In this section we shall prove the existence of globally regular solutions of semilinear wave equations with small initial data. Given a function u on a cone $K(z_0)$ we denote its energy by

$$e(u) = \frac{1}{2}(|u_t|^2 + |\nabla u|^2) + \frac{1}{4}u^4$$

and by

$$E(u : D(t : z_0)) = \int_{D(t:z_0)} e(u)\,dx$$

its energy on the space-like section $D(t : z_0)$. Let $x = y + x_0$. We denote by

$$d_{z_0}(u) = \frac{1}{2}|\hat{y}u_t - \nabla u|^2 + \frac{1}{4}u^4$$

the energy density of u tangent to $M(z_0)$. The following Hardy's inequalities are useful to prove the regular solutions of semilinear wave equations.

LEMMA 2.1. *Suppose $u \in L^4(B_R)$ possesses a weak radial derivative $u_r = \hat{x} \cdot \nabla u \in L^2(B_R)$. Then with an constant C_0 independent on ρ and R for all $0 \le \rho < R$ the following holds:*

$$\frac{3}{4}\int_{B_R \setminus B_\rho} \frac{|u(x)|^2}{|x|^2}\, dx \le \int_{B_R \setminus B_\rho} |u_r|^2\, dx + \frac{1}{2R}\int_{S_R} |u|^2\, do. \tag{2.1}$$

$$\int_{B_R} \frac{|u(x)|^2}{|x|^2}\, dx \le C_0 \left\{ \int_{B_R} |u_r|^2\, dx + \left(\int_{B_R} u^4\, dx \right)^{1/2} \right\} \tag{2.2}$$

$$\int_{S_R} u^3\, do \le C_0 \left\{ \left(\int_{B_R} u^4\, dx \right)^{1/2} \left(\int_{B_R} u_r^2\, dx \right)^{1/2} + \left(\int_{B_R} u^4\, dx \right)^{3/4} \right\} \tag{2.3}$$

Proof. The equality

$$(\sqrt{r}u)_r = \sqrt{r}u_r + \frac{u}{2\sqrt{r}}, \quad r = |x|$$

implies

$$u_r^2 = \left| \frac{1}{\sqrt{r}}(\sqrt{r}u)_r - \frac{1}{2r}u \right|^2 \ge \frac{u^2}{4r^2} - \frac{1}{2r^2}(ru^2)_r. \tag{2.4}$$

Integrating (2.4) over $B_R \setminus B_\rho$, we have

$$\frac{1}{4}\int_{B_R \setminus B_\rho} \frac{u^2}{r^2}\, dx \le \int_{B_R \setminus B_\rho} u_r^2\, dx + \frac{1}{2}\int_{B_R \setminus B_\rho} \left\{ \nabla \cdot \left(\frac{u^2}{r^2} x \right) - \frac{u^2}{r^2} \right\} dx.$$

Therefore, the divergence theorem yields (2.1). Note that

$$\left(r^3 \int_{S_1} u^3(r\xi)\, d\xi \right)_r = 3r^2 \int_{S_1} u^3(r\xi)\, d\xi + 3r^3 \int_{S_1} u^2(r\xi) u_r(r\xi)\, d\xi. \tag{2.5}$$

Integrating (2.5) from 0 to R, we have

$$\begin{aligned} \int_{S_R} u^3\, do =& 3\int_{B_R} \frac{u^3}{r}\, dx + 3\int_{B_R} u^2 u_r\, dx \\ \le& 3\left(\int_{B_R} u^4\, dx \right)^{1/2} \left\{ \left(\int_{B_R} \frac{u^2}{r^2}\, dx \right)^{1/2} + \left(\int_{B_R} u_r^2\, dx \right)^{1/2} \right\}. \end{aligned}$$

Since

$$\int_{S_R} u^2\,do \le R(4\omega_4)^{1/3}\left(\int_{S_R} u^3\,do\right)^{2/3},$$

we have

$$\begin{aligned}
\frac{3}{4}\int_{B_R}\frac{u^2}{r^2}\,dx &\le \int_{B_R} u_r^2\,dx + \frac{1}{2R}\int_{S_R} u^2\,do\\
&\le \int_{B_R} u_r^2\,dx + \left(\frac{\omega_4}{2}\right)^{1/3}\left(\int_{S_R} u^3\,do\right)^{2/3}\\
&\le \int_{B_R} u_r^2\,dx + \left(\frac{9\omega_4}{2}\right)^{1/3}\left(\int_{B_R} u^4\,dx\right)^{1/3}\\
&\quad\left\{\left(\int_{B_R}\frac{u^2}{r}\,dx\right)^{1/2} + \left(\int_{B_R} u_r^2\,dx\right)^{1/2}\right\}^{2/3}\\
&\le \int_{B_R} u_r^2\,dx + \left(\frac{3^5\omega_4}{2^4}\right)^{1/3}\left(\int_{B_R} u^4\,dx\right)^{1/2}\\
&\quad + \left(\frac{\omega_4}{6}\right)^{1/3}\left\{\int_{B_R}\frac{u^2}{r^2}\,dx + \int_{B_R} u_r^2\,dx\right\}.
\end{aligned}$$

This implies (2.2). Finally, using (2.2), we have

$$\begin{aligned}
&\int_{S_R} u^3\,do\\
\le& 3\left(\int_{B_R} u^4\,dx\right)^{1/2}\left\{\left(\int_{B_R}\frac{u^2}{r^2}\,dx\right)^{1/2} + \left(\int_{B_R} u_r^2\,dx\right)^{1/2}\right\}\\
\le& C\left\{\left(\int_{B_R} u^4\,dx\right)^{1/2}\left(\int_{B_R} u_r^2\,dx\right)^{1/2} + \left(\int_{B_R} u^4\,dx\right)^{3/4}\right\}. \qquad ////
\end{aligned}$$

Note that if $u = u(x,t)$ is a solution of (1.1), then $u(x,-t)$is also a solution of (1.1). Since the semilinear wave equation is conformally invariant, the solution is translation invariant in t.

Let $\bar z = (\bar x, \bar t)$ be given and suppose u is a C^2-solution of (1.1) on the deleted backward light cone $K_0'(\bar z) = K_0(\bar z)\backslash\{\bar z\}$. In order to prove that u can be extended to a global solution of (1.1) and (1.2), it suffices to show that for any $\bar z$ as above

$$\bar m = limsup_{\substack{z_0\to\bar z\\ z_0\in K(\bar z), z_0\neq\bar z}} |u(z_0)| < \infty.$$

We may assume that $\bar m = sup_{K_0(\bar z)}|u|$.

LEMMA 2.2. *Suppose $u \in C^2(K_0'(\bar{z}))$ solve (1.1) and (1.2). Then for any $0 \le t < s < \bar{t}$ there holds*

$$E(u : D(s, \bar{z})) + \frac{1}{\sqrt{2}} \int_{M_t^s(\bar{z})} d_{\bar{z}}(u)\, do = E(u : D(t : \bar{z})) \le E_0$$

Proof. Integrating the identity

$$e(u)_t - \operatorname{div}(u_t \nabla u) = e(u)_t - \operatorname{div} \vec{p}(u) = 0$$

over a cone K_t^s of the positive light cone and using the identity

$$e(u) - \frac{x}{|x|} \cdot \vec{p}(u) = d_{z_0}(u),$$

we obtain the result. ////

By Lemma 2.2, for any fixed $\bar{z}$ the energy $E(u : D(s, \bar{z}))$ is a monotone decreasing function of $s \in [0, \bar{t})$ and hence converges to a limit as $s \nearrow \bar{t}$. It follows that

$$\int_{M_t^s(\bar{z})} d_{\bar{z}}(u)\, do \to 0 \qquad \text{as } s, t \nearrow \bar{t} \tag{2.6}$$

In Section 1, we had a decomposition of the solution of (1.1) and (1.2) as

$$u = u_L + u_N,$$

where $y = x - x_0$, and u_L and u_N are defined as in (1.7) and (1.9) respectively. Since we are interested in points z_0 such that $|u(z_0)| \to \bar{m}$ as $z_0 \to \bar{z}$, we need only consider points z_0 satisfying $|u(z_0)| = max_{K_0(z_0)}|u| = m_0$. Thus, and splitting integration over $M_0^T(z_0)$ and $M_T(z_0)$ for suitable T, from Hölder's inequality we obtain

$$\begin{aligned} m_0 =& |u(z_0)| \\ \le& C + \frac{m_0}{\sqrt{2}\omega_4} \int_{M_T(z_0)} \frac{u^2(z)}{|z - z_0|^2}\, do + \frac{1}{\sqrt{2}\omega_4} \int_{M_0^T(z_0)} \frac{u^3(z)}{|z - z_0|^2}\, do. \end{aligned} \tag{2.7}$$

By Lemma 2.2 the last term is bounded by $C|t_0 - T|^{-1} E_0^{\frac{3}{4}}$. Thus to establish our main result, it suffices to show that for any $\bar{z} = (\bar{x}, \bar{t})$ there exists $T < \bar{t}$ such that

$$limsup_{\substack{z_0 \to \bar{z} \\ z_0 \in K(\bar{z})}} \int_{M_T(\bar{z})} \frac{u^2(z)}{|z - z_0|^2} do < \sqrt{2}\omega_4. \tag{2.8}$$

This observation and Hardy's inequality gives

THEOREM 2.3. *If $u \in C^2([0,T) \times R^4)$ is a solution of (1.1) and (1.2),then there exists a constant $\epsilon_0 > 0$ such that for any $u_0 \in C^4(R^4), u_1 \in C^3(R^4)$ with*

$$E_0 = \int_{R^4} \Big(\frac{1}{2}(|u_1|^2 + |\nabla u_0|^2) + \frac{1}{4}|u_0|^4\Big) dx < \epsilon_0,$$

(1.1) and (1.2) admit a global C^2 solution.

Proof. Let $v(y) = u(x_0 + y, t_0 - |y|)$. Then by Lemma 2.1 we have

$$\begin{aligned}\int_{M_T(z_0)} \frac{|u|^2(z)}{|z - z_0|^2}\, do &= \frac{1}{\sqrt{2}} \int_{B_{t_0 - T}(0)} \frac{|v(y)|^2}{|y|^2}\, dy \qquad (2.9)\\ &\le C \int_{B_{t_0-T}(0)} |\nabla v|^2\, dy + C \left(\int_{B_{t_0-T}(0)} |u|^4\, dy \right)^{1/2}\\ &\le C \int_{M_T(z_0)} d_{z_0}(u)\, do + C \left(\int_{M_T(z_0)} d_{z_0}(u)\, do \right)^{1/2}\\ &\le C(E_0 + E_0^{1/2}).\end{aligned}$$

Letting T=0, the theorem holds from (2.7). ////

Since $t = 0$ no longer plays a distinguished role in the following, we may shift coordinates so that $\bar{z} = (0,0)$ and thus in the sequel we may assume that u is a C^2 solution of (1.1) on $K_{t_1} \backslash \{(0,0)\}$ for some $t_1 < 0$.

3. Some Estimates for the Large Initial Data

In this section, we introduce the multiplier $tu_t + x \cdot \nabla u + \frac{3}{2} u$ to drive the following identity

$$\partial_t Q_d - \mathrm{div} P_d + R_d = 0, \qquad (3.1)$$

where

$$\begin{aligned}
Q_d =& te(u) + x\cdot \vec{p}(u) + \frac{3}{2}uu_t \\
=& \frac{1}{4}(t-r)(u_t-u_r)^2 + \frac{1}{4}(t+r)(u_t+u_r)^2 \\
& + \frac{1}{2}t|\nabla u - u_r\hat{x}|^2 + \frac{1}{4}tu^4 + \frac{3}{2}uu_t \\
=& Q_0 + \frac{3}{2}uu_t, \\
P_d =& t\vec{p}(u) + xl(u) + (x\cdot\nabla u)\nabla u + \frac{3}{2}u\nabla u, \\
R_d =& \frac{1}{4}u^4.
\end{aligned}$$

The identity (3.1) is equivalent to the identity

$$\begin{aligned}
& t\left\{\frac{d}{dt}(e(u) + \frac{x}{t}\cdot\vec{p}(u) + \frac{3}{2t}uu_t + \frac{3}{4t^2}u^2)\right. \\
& \left. -\mathrm{div}(\vec{p}(u) + \frac{x}{t}l(u) + \frac{1}{t}(x\cdot\nabla u)\nabla u + \frac{3}{2t}u\nabla u)\right\} \\
& + e(u) + \frac{1}{t}x\cdot\vec{p}(u) + \frac{3}{2t^2}u^2 + R_d = 0.
\end{aligned} \tag{3.2}$$

LEMMA 3.1. *There exists a sequence of numbers $t_l \nearrow 0$ such that*

$$\frac{1}{|t_l|}\int_{D(t_l)} uu_t\,dx \le o(1), \tag{3.3}$$

where $o(1) \to 0$ as $l \to \infty$.

Proof. Consider $u_l(x,t) = 2^{-l}u(2^{-l}x, 2^{-l}t)$, $\quad l \in N$, satisfying (1.1) with

$$E(u_l; D(t)) = E(u; D(2^{-l}t)) \le E_0.$$

Note that

$$\int_{M_{t_1}} d_0(u_l)dz \to 0 \tag{3.4}$$

as $l \to \infty$. Now, if

$$\int_{D(t_1)} u_l^2 \, dx \to 0 \quad (l \to \infty), \tag{3.5}$$

letting $t_l = 2^{-l} t_1$ we have an estimate

$$\begin{aligned} \frac{1}{|t_l|} \int_{D(t_l)} u_t u \, dx &\leq \left(\int_{D(t_l)} |u_t|^2 \, dx \right)^{1/2} \left(\frac{1}{|t_l|^2} \int_{D(t_l)} u^2 \, dx \right)^{1/2} \\ &\leq 2E(u; D(t_l))^{1/2} \left(\frac{1}{|t_l|^2} \int_{D(t_l)} u_l^2 \, dx \right)^{1/2} \end{aligned} \tag{3.6}$$

which converges to 0 as $l \to \infty$. Otherwise, there exist a positive constant C_1 and a sequence Λ of numbers $l \to \infty$ such that

$$\lim_{\substack{l \to \infty \\ l \in \Lambda}} \int_{D(t_1)} u_l^2 \, dx \geq C_1. \tag{3.7}$$

For any $s \in [t_1, 0)$, by Hölder's inequality

$$\begin{aligned} \int_{D(s)} u_l^2 \, dx &\leq \left(\omega_4 |s|^4 \right)^{1/2} \left(\int_{D(s)} u_l^4 \, dx \right)^{1/2} \\ &\leq C E_0^{1/2} s^2. \end{aligned} \tag{3.8}$$

Choose $s = s_1 < 0$ such that the latter is $\leq C_1$. Then by (3.4) we have

$$\begin{aligned} 2 \int_{K_{t_1}^{s_1}} (u_l)_t u_l \, dz &= \int_{K_{t_1}^{s_1}} \left(|u_l|^2 \right)_t \, dz \\ &= \int_{D(s_1)} |u_l|^2 \, dx - \int_{D(t_1)} |u_l|^2 \, dx + \frac{1}{\sqrt{2}} \int_{M_{t_1}^{s_1}} |u_l|^2 \, do \\ &\leq o(1) \to 0 \quad (l \to \infty, \quad l \in \Lambda). \end{aligned}$$

We conclude that for suitable numbers $s_l \in [t_1, s_1], t_l = 2^{-l} s_l, l \in \Lambda$, we have

$$\begin{aligned} \frac{2}{|t_l|} \int_{D(t_l)} u_t u \, dx &= \frac{2}{|s_l|} \int_{D(s_l)} (u_l)_t u_l \, dx \\ &\leq o(1) \to 0 \quad (l \to \infty, \quad l \in \Lambda). \end{aligned}$$

Relabelling, we obtain a sequence $\{t_l\}_{l \in N}$, as desired. ////

LEMMA 3.2. *For any $l \in N$ there holds*

$$\frac{1}{4|t_l|}\int_{K_{t_l}} |u|^4\,dxdt + \int_{D(t_l)}\Big\{e(u) + \frac{x}{t}\cdot\vec{p}(u)\Big\}\,dx \le o(1) \to 0 \qquad (3.9)$$

as $l \to \infty$.

Proof. For $s \in [t_l, 0)$, we integrate (3.1) over $K^s_{t_l}$ to obtain

$$\begin{aligned}
0 &= \int_{k^s_{t_l}} \{(Q_d)_t - \mathrm{div} P_d + R_d\}dxdt \\
&= \int_{D(s)} Q_d dx - \int_{D(t_l)} Q_d dx + \frac{1}{\sqrt{2}}\int_{M^s_{t_l}} (Q_d - x\cdot P_d)do + \int_{K^s_{t_l}} R_d dxdt \\
&= \int_{D(s)}\Big\{s(e(u) + \frac{1}{s}x\cdot\vec{p}(u)) + \frac{3}{2}u_t u\Big\}dx \\
&\quad + \frac{1}{\sqrt{2}}\int_{M^s_{t_l}}\Big\{te(u) + x\cdot\vec{p}(u) + \frac{3}{2}uu_t - x\cdot P_d\Big\}\,ds \\
&\quad - |t_l|\int_{D(t_l)}\Big\{e(u) + \frac{1}{|t_l|}x\cdot\vec{p}(u)\Big\}\,dx + \int_{K^s_{t_l}} R_d\,dxdt - \frac{3}{2}\int_{D(t_l)} uu_t\,dx.
\end{aligned}$$

Now, $e(u) + \frac{1}{t}x\cdot\vec{p}(u)$ is dominated by the energy density $e(u)$. Therefore, using Hölder inequality as in (3.6) and (3.8), the first term is of order $|s|$ and hence vanishes as $s \to 0$. Moreover, on M_{t_l} we have

$$\begin{aligned}
&\Big|te(u) + x\cdot\vec{p}(u) + \frac{3}{2}uu_t - \hat{x}\cdot P_d\Big| \\
=&|t|\Big|e(u) + \frac{1}{t}x\cdot\vec{p}(u) + \frac{3}{2t}uu_t - \hat{x}\cdot\vec{p}(u) - l(u) \\
&\quad - |\hat{x}\cdot\nabla u|^2 - \frac{3u}{2t^2}x\cdot\nabla u\Big| \\
=&|t|\Big||\nabla u|^2 - |\hat{x}\cdot\nabla u|^2 + \frac{1}{2}u^4 - \frac{3}{2t^2}u(tu_t + x\cdot\nabla u)\Big| \\
\le&|t_l|\Big(2d_0(u) + \frac{|u|^2}{t^2}\Big).
\end{aligned}$$

Hence by (2.1) and Lemma 2.1 the second term is of order $o(1)|t_l|$, where $o(1) \to 0$ as $l \to \infty$. Thus, by Lemma 3.1 we have

$$\begin{aligned}\frac{1}{|t_l|}\int_{K^*_{t_l}} R_d\,dxdt + \int_{D(t_l)}\Big(e(u) + \frac{1}{|t_l|}x\cdot\vec{p}(u)\Big)\,dx \\ \le \frac{3}{2|t_l|}\int_{D(t_l)} uu_t\,dx + o(1) \\ \le o(1) \to 0 \quad (l\to\infty)\end{aligned}$$

which is the desired conclusion. ////

LEMMA 3.3. *There exists a sequence of numbers $\bar{t}_l \nearrow 0$ such that the conclusion of Lemma 3.1 holds for $(\bar{t}_l)$ which in addition we have*

$$2 \le \frac{\bar{t}_l}{\bar{t}_{l+1}} \le 4 \tag{3.11}$$

for all $l \in N$.

Proof. First observe that by Hölder's inequality and by Lemma 3.2 we have for any $m \in N$

$$\begin{aligned}&\int_{D(t_m)} \frac{|u|^2}{|t|^2}\,dx \\ &\le \left(\int_{D(t_m)} \frac{1}{|t|^4}\,dx\right)^{1/2}\left(\int_{D(t_m)} |u|^4\,dx\right)^{1/2} \\ &= \left(\frac{1}{|t_m|^4}\omega_4|t_m|^4\right)^{1/2}\left(\int_{D(t_m)} |u|^4\,dx\right)^{1/2} \\ &\le C\omega_4^{1/2}\int_{D(t_m)}\left\{e(u) + \frac{1}{t}(x\cdot\nabla u)u_t\right\}dx \to 0 \quad (m\to\infty),\end{aligned}$$

where $\{t_m\}$ is determined in Lemma 3.1. From the identity (3.2) we have

$$\begin{aligned}&\frac{d}{dt}\left\{e(u) + \frac{x}{t}\cdot\vec{p}(u) + \frac{3}{2t}uu_t + \frac{3}{4t^2}u^2\right\} \\ &- \operatorname{div}\left\{\vec{p}(u) + \frac{x}{t}l(u) + \frac{1}{t}(x\cdot\nabla u)\nabla u + \frac{3}{2t}u\nabla u\right\} \\ &+ \frac{1}{t}\left\{e(u) + \frac{1}{t}x\cdot\vec{p}(u) + \frac{3}{2t^2}u^2 + R_d\right\} = 0.\end{aligned} \tag{3.12}$$

Integrate (3.12) over the cone $K_{t_l}^{t_m}$ for $m \geq l$ to obtain

$$\begin{aligned} &\int_{D(t_l)} \left\{ e(u) + \frac{x}{t} \cdot \vec{p}(u) + \frac{3}{2t} u u_t + \frac{3}{4t^2} u^2 \right\} dx \qquad (3.13) \\ &+ \int_{K_{t_l}^{t_m}} \frac{1}{t} \left\{ e(u) + \frac{x}{t} \cdot \vec{p}(u) + + \frac{3}{2t^2} u^2 + R_d \right\} dx dt \\ = &\int_{D(t_m)} \left\{ e(u) + \frac{x}{t} \cdot \vec{p}(u) + \frac{3}{4t^2} u^2 \right\} dx \\ &+ \int_{M_{t_l}^{t_m}} \Big\{ e(u) + \frac{x}{t} \cdot \vec{p}(u) + \frac{3}{2t} u u_t + \frac{3}{4t^2} u^2 \\ &- \frac{x}{t} \cdot \Big(\vec{p}(u) + \frac{x}{t} l(u) + (\frac{x}{t} \cdot \nabla u) \nabla u + \frac{3}{2t} u \nabla u \Big) \Big\} do. \end{aligned}$$

By the preceding remark the first term on the right (3.13) vanishes as we let $m \to \infty$, while by (3.1) the integral over $M_{t_l}^{t_m}$ becomes arbitrarily small as $m \geq l \to \infty$. Finally, by Lemma 3.1, we have

$$\int_{D(t_l)} \frac{1}{t} u u_t \, dx = -\frac{1}{|t_l|} \int u u_t \, dx \geq o(1) \to 0 \quad (l \to \infty).$$

All remaining term being non-negative, we thus obtain the estimates

$$\int_{K_{t_l}} \frac{u^2}{|t|^3} \, dx dt = \int_{t_l}^{0} \left(\frac{1}{|t|} \int_{D(t)} \frac{u^2}{|t|^2} \, dx \right) dt \leq o(1) \to 0 \quad \text{as} \quad l \to \infty.$$

Hence for any $\bar{t} \in [\frac{t_l}{2}, 0)$ there also holds

$$o(1) \geq \frac{1}{\bar{t}} \int_{2\bar{t}}^{\bar{t}} \left(\int_{D(t)} \frac{u^2}{|t|^2} \, dx \right) dt \geq \inf_{2\bar{t} \leq t \leq \bar{t}} \int_{D(t)} \frac{u^2}{|t|^2} \, dt$$

where $o(1) \to 0$ if $l \to \infty$. Now to construct the sequence $\{\bar{t}_l\}$, choose $\bar{t}_1 = t_1$ and proceed by induction. Suppose $\bar{t}_l$, $l = 1, ..., L$, have been defined already. Let $\bar{t}_{L+1} \in [\frac{\bar{t}_L}{2}, \frac{\bar{t}_L}{4})$ be chosen such that

$$\int_{D(\bar{t}_{L+1})} \frac{u^2}{|t|^2} \, dx \leq 2 \inf_{\frac{\bar{t}_L}{2} \leq t \leq \frac{\bar{t}_L}{4}} \int_{D(t)} \frac{u^2}{|t|^2} \, dx.$$

Clearly, this procedure yields a sequence $\{\bar{t}_l\}$ such that $2 \le \frac{\bar{t}_l}{\bar{t}_{l+1}} \le 4$ for all l and we have

$$\int_{D(\bar{t}_l)} \frac{u^2}{|t|^2}\, dx \to 0 \quad (l \to \infty).$$

Then by (3.4) we have

$$\frac{1}{|t_l|} \int_{D(\bar{t}_l)} u u_t \, dx \to 0 \quad (l \to \infty)$$

concluding the proof. ////

In the sequel to simplify notation we shall assume that $t_l = \bar{t}_l$ for all l, initially.

4. Globally Regular Solutions for the General Data

In this section we shall prove the Theorem 0.1. Fix $z_0 = (x_0, t_0) \in K\backslash\{0\}$ arbitrary. Let $y = x - x_0, \hat{y} = \frac{y}{|y|}, \hat{x} = \frac{x}{|x|}$. Divide (3.2) by t and then for $s > t_0$ integrate over $K^s_{t_l}\backslash K(z_0)$ to obtain the relation

$$\begin{aligned}
0 =& \int_{D(s)} \Big\{ e(u) + \frac{1}{t} x \cdot \vec{p}(u) + \frac{3}{2t} u u_t + \frac{3}{4t^2} u^2 \Big\}\, dx \\
&- \int_{D(t_l)\backslash D(t_l : z_0)} \Big\{ e(u) + \frac{1}{t} x \cdot \vec{p}(u) + \frac{3}{2t} u u_t + \frac{3}{4t^2} u^2 \Big\}\, dx \\
&+ \frac{1}{\sqrt{2}} \int_{M^s_{t_l}} \Big\{ e(u) + \frac{1}{t} x \cdot \vec{p}(u) + \frac{3}{2t} u u_t + \frac{3}{4t^2} u^2 - \hat{x} \cdot P \Big\}\, do \\
&- \frac{1}{\sqrt{2}} \int_{M_{t_l}(z_0)} \Big\{ e(u) + \frac{1}{t} x \cdot \vec{p}(u) + \frac{3}{2t} u u_t + \frac{3}{4t^2} u^2 - \hat{y} \cdot P \Big\}\, do \\
&+ \int_{K^s_{t_l}\backslash K(z_0)} \frac{1}{t} \Big\{ e(u) + \frac{1}{t} x \cdot \vec{p}(u) + \frac{3}{2t^2} u^2 + R_d \Big\}\, dx dt \\
=& I + II + III + IV + V,
\end{aligned}$$

where $P = \vec{p}(u) + \frac{x}{t} l(u) + (\frac{1}{t} x \cdot \nabla u) \nabla u + \frac{3}{2t} u \nabla u = \frac{1}{t} P_d$.

By Hölder's inequality, (3.6), (3.8) and Lemma 3.2 the first term $I \to 0$ if we choose $s = t_k$ with $k \to \infty$. Similarly, $II \to 0$ if $l \to \infty$.

By Lemma 3.2 also $III \to 0$ as $l \to \infty$. Finally $V \leq 0$. Thus we obtain the estimate for any $z_0 \in K \backslash \{0\}$.

$$\int_{M_{t_l}(z_0)} \left\{ e(u) + \frac{1}{t} x \cdot \vec{p}(u) + \frac{3}{2t} u u_t + \frac{3}{4t^2} u^2 - \hat{y} \cdot P \right\} do \leq o(1) \to 0 \quad (4.1)$$

as $l \to \infty$, with error term $o(1)$ independent of z_0.

In order to bound (2.8) we shall use (4.1). Let $r = |x|$; then we may rewrite

$$\begin{aligned} A :=& e(u) + \frac{1}{t} x \cdot \vec{p}(u) + \frac{3}{2t} u u_t + \frac{3}{4t^2} u^2 - \hat{y} \cdot P \\ =& \frac{1}{2}\big(1 - \frac{r}{t} \hat{x} \cdot \hat{y}\big)|u_t|^2 + \big(1 + \frac{r}{t} \hat{x} \cdot \hat{y}\big)\big(\frac{1}{2}|\nabla u|^2 + \frac{1}{4}|u|^4\big) \\ &+ \frac{3}{2t}(u_t - \hat{y} \cdot \nabla u) u + \frac{r}{t}(u_t - \hat{y} \cdot \nabla u) \hat{x} \cdot \nabla u - u_t \hat{y} \cdot \nabla u + \frac{3}{4t^2} u^2. \end{aligned}$$

Introducing $u_\sigma = \hat{y} \cdot \nabla u, \alpha = \hat{x} - \hat{y}(\hat{y} \cdot \hat{x}), |\alpha| u_\alpha = \alpha \cdot \nabla u, \Omega u = \nabla u - \hat{y} u_\sigma$, we have

$$\begin{aligned} A :=& \frac{1}{2}\big(1 - \frac{r}{t} \hat{x} \cdot \hat{y}\big)(u_t - u_\sigma)^2 + \big(1 + \frac{r}{t} \hat{x} \cdot \hat{y}\big)\big(\frac{1}{2}|\Omega u|^2 + \frac{1}{4}|u|^4\big) \\ &+ \frac{3}{2t}(u_t - u_\sigma) u + \frac{r}{t}|\alpha| u_\alpha (u_t - u_\sigma) + \frac{3}{4t^2} u^2 \end{aligned}$$

Now let $\hat{x} \cdot \hat{y} = \cos\delta$, $|\alpha| = \sin\delta$ and let $u_\rho = \frac{1}{\sqrt{2}}(u_t - u_\sigma)$. Then we have

$$\begin{aligned} A =& \big(1 - \frac{r}{t}\cos\delta\big)|u_\rho|^2 + \big(1 + \frac{r}{t}\cos\delta\big)\big(\frac{1}{2}|\Omega u|^2 + \frac{1}{4}|u|^4\big) \\ &+ \frac{r}{t}\sqrt{2}|\sin\delta| u_\rho u_\alpha + \frac{3}{\sqrt{2}t} u u_\rho + \frac{3}{4t^2} u^2 \\ =& A_0 + \frac{3}{\sqrt{2}t} u u_\rho + \frac{3}{4t^2} u^2. \end{aligned} \quad (4.2)$$

Note that if we estimate $|u_\alpha| \leq |\Omega u|$, then we have

$$\begin{aligned} A_0 \geq& \big(1 - \frac{r}{t}\cos\delta\big)|u_\rho|^2 + \big(1 + \frac{r}{t}\cos\delta\big)\big(\frac{1}{2}|u_\alpha|^2 + \frac{1}{4}|u|^4\big) \\ &+ \frac{r}{t}\sqrt{2}|\sin\delta| u_\rho u_\alpha \\ =& \big(1 + \frac{r}{t}\big)\big(|u_\rho|^2 + \frac{1}{2}|u_\alpha|^2\big) - \frac{r}{2t}\big(\sqrt{2}\sqrt{1 + \cos\delta}\, u_\rho - \sqrt{1 - \cos\delta}\, u_\alpha\big)^2 \\ &+ \frac{1}{4}\big(1 + \frac{r}{t}\cos\delta\big)|u|^4 \geq 0 \end{aligned} \quad (4.3)$$

on $M_{t_l}(z_0)$.

Now for any $\epsilon > 0$ there exists a constant $C = C(\epsilon)$ such that for any $z_0 \in K$ and any $z \in M^{Ct_0}(z_0)$ we may estimate

$$-\frac{r}{t}\sqrt{2}|\sin\delta| \leq \epsilon, \qquad -\frac{r}{t}\cos\delta \geq \frac{1}{2}.$$

In fact, for $z = (x,t) \in M^{Ct_0}(z_0)$ we have

$$\big||x| - |y|\big| \leq |y - x| = |x_0| \leq |t_0| \leq \frac{|t - t_0|}{C-1} = \frac{|y|}{C-1}.$$

Hence

$$\hat{x}\cdot\hat{y} = \cos\delta \geq 1 - |\hat{y} - \hat{x}| \geq 1 - 2\frac{|x_0|}{|y|} \geq 1 - \frac{2}{C-1}$$

while

$$1 \geq -\frac{r}{t} = \frac{|y|}{|t-t_0|}\frac{|t-t_0|}{|t|}\frac{|x|}{|y|} \geq (1 - \frac{1}{C})(1 - \frac{1}{C-1}).$$

This yields the following estimate.

LEMMA 4.1. *For any $\epsilon > 0$, any $z_0 \in K$, letting $C = C(\epsilon)$ be determined as above for $t_k \leq Ct_0$ we have*

$$\int_{M_{t_l}^{t_k}(z_0)} A\,do \geq \frac{1}{2}\int_{M_{t_l}^{t_k}(z_0)} |u_\rho|^2\,do - \epsilon E_0$$

Proof. we note first that

$$\left|\frac{\sqrt{2}u_\rho u}{t}\right| \leq |u_\rho|^2 + \frac{3}{4t^2}u^2.$$

Hence by (4.2) and our choice of $C(\epsilon)$, for $z \in M_{t_l}^{Ct_0}(z_0)$ we have

$$\begin{aligned} A &\geq \frac{1}{2}|u_\rho|^2 - \epsilon|u_\rho u_\alpha| \\ &\geq \frac{1}{2}|u_\rho|^2 - \epsilon e(u), \end{aligned}$$

which proves the lemma. ////

Note that u_ρ may be interpreted as a tangential derivative along $M(z_0)$. In fact, let Φ be the map

$$\Phi : y \to (x_0 + y, t_0 - |y|) \tag{4.4}$$

and let

$$v(y) = u(\Phi(y)) \tag{4.5}$$

wherever the latter is defined. Then the radial derivative v_s of v is given by

$$v_s = \hat{y} \cdot \nabla v = u_\sigma - u_t = -\sqrt{2} u_\rho. \tag{4.6}$$

LEMMA 4.2. *For any $z_0 \in K$ and any $C \geq 0$ there holds*

$$\int_{M_{(1+C)t_0}(z_0)} \frac{u_\rho u}{t}\, do \geq (1 + log(1+C))o(1),$$

where $o(1) \to 0$ if $(1+C)t_0 \geq t_l$ and $l \to \infty$.

Proof. Introducing y as new variable, via (4.4),(4.5) we have

$$\begin{aligned}\int_{M_{(1+C)t_0}(z_0)} \frac{u_\rho u}{t}\, do &= \int_{B_{C|t_0|}} \frac{v_s v}{|y| - t_0}\, dy \\ &= \int_{S_1} \left(\int_0^{C|t_0|} \frac{v_s v}{s - t_0} s^2\, ds \right) do.\end{aligned}$$

Integrating by parts, this gives

$$\begin{aligned}&\int_{S_1} \left(\int_0^{C|t_0|} \frac{s^2}{s-t_0} \frac{\partial}{\partial s}\left(\frac{v^2}{2}\right) ds \right) do \\ &= \int_{S_1} \int_0^{C|t_0|} \left\{ -\frac{v^2 s}{s - t_0} + \frac{v^2 s^2}{2(s-t_0)^2} \right\} ds\,do + \frac{1}{2(1+C)|t_0|} \int_{S_{C|t_0|}} v^2\, do \\ &\geq - \int_{B_{C|t_0|}} \frac{v^2}{|y|(|y| - t_0)}\, dy \\ &= -\frac{1}{\sqrt{2}} \int_{M_{(1+C)t_0}(z_0)} \frac{u^2}{s(s-t_0)}\, do(x,t) \\ &= -\int_0^{C|t_0|} \frac{1}{s - t_0} \left(\frac{1}{s} \int_{\partial D(s - t_0 : z_0)} u^2\, do(x) \right) ds\end{aligned}$$

Now by Lemma 2.1

$$\begin{aligned}
&\left(\frac{1}{s}\int_{\partial D(s-t_0:z_0)} u^2\,do\right)^{3/2}\\
\leq & C\int_{\partial D(s-t_0:z_0)} u^3\,do\\
\leq & C\left\{\left(\int_{\partial D(s-t_0:z_0)} u^4\,dx\right)^{1/4}+\left(\int_{\partial D(s-t_0:z_0)} |\nabla u|^2\,dx\right)^{1/2}\right\}\\
&\left(\int_{\partial D(s-t_0:z_0)} u^4\,dx\right)^{1/2}\\
\leq & CE_0\left(\int_{\partial D(s-t_0:z_0)} u^4\,dx\right)^{1/2}
\end{aligned}$$

Hence

$$\int_{M_{(1+C)t_0}(z_0)} \frac{u_\rho u}{t}\,do \geq -C\int_{(1+C)t_0}^{t_0} \frac{1}{|t|}\left(\int_{\partial D(s-t_0:z_0)} u^4\,dx\right)^{1/3} dt$$

with $C = C(E_0)$. By Lemma 3.2 the latter can be controlled as follows. Let $k, K \in N$ be determined such that

$$t_k \leq (1+C)t_0 < t_{k+1} \leq t_K \leq t_0 < t_{K+1}.$$

Note that by Lemma 3.3

$$1 + C \geq \frac{t_{k+1}}{t_K} \geq 2^{K-(k+1)}$$

whence

$$K - k \leq 1 + \log_2(1 + C).$$

We have the estimate

$$\begin{aligned}
I = & \int_{(1+C)t_0}^{t_0} \frac{1}{|t|}\left(\int_{D(t)} u^4\,dx\right)^{1/3} dt\\
\leq & \sum_{i=k}^{K}\int_{t_i}^{t_{i+1}} \frac{1}{|t|}\left(\int_{D(t)} u^4\,dx\right)^{1/3} dt.
\end{aligned}$$

By Hölder's inequality,

$$I \le C \sum_{i=k}^{K} \frac{|t_i - t_{i+1}|^{2/3}}{|t_{i+1}|} \left(\int_{K_{t_i}^{t_{i+1}}} u^4 \, dz \right)^{1/3}$$

and by Lemma 3.3

$$I \le C \sum_{i=k}^{K} \Big(\frac{1}{t_i} \int_{K_{t_i}} u^4 \, dz \Big)^{1/3}.$$

Finally, use Lemma 3.2 to see that

$$I \le (K - k + 1) o(1) \le (1 + \log(1 + C)) o(1)$$

where $o(1) \to 0$ if $(1 + C) t_0 \ge t_l$ and $l \to \infty$. ////

Combing Lemma 4.1 and Lemma 4.2 it follows that for any $\epsilon > 0$, if we choose $t_k \le C(\epsilon) t_0 < t_{k+1}$, we can estimate

$$\begin{aligned} o(1) &\ge \int_{M_{t_l}(z_0)} A \, do \\ &\ge \frac{1}{2} \int_{M_{t_l}^{t_k}(z_0)} |u_\rho|^2 \, do - \epsilon E_0 \\ &\quad + \int_{M_{t_l}(z_0)} A_0 \, do - o(1) \left(1 + log(1 + C(\epsilon))\right), \end{aligned} \tag{4.7}$$

where $o(1) \to 0$ as $l \to \infty$. To estimate A_0 on $M_{t_k}(z_0)$ now introduce the new angle δ_0, where $|x_0| = r_0, \hat{x}_0 = \dfrac{1}{r_0} x_0, \hat{x}_0 \cdot \hat{y} = \cos \delta_0$. Again let $y = x - x_0$ and $|y| = \sigma = |t - t_0|$. With this notation

$$\begin{aligned} r \hat{x} \cdot \hat{y} &= x \cdot \hat{y} = y \cdot \hat{y} + x_0 \cdot \hat{y} \\ &= \sigma + r_0 \cos \delta_0, \end{aligned}$$

$$\begin{aligned} |\alpha| &= \left| \frac{x - (x \cdot \hat{y}) \hat{y}}{r} \right| = \left| \frac{x_0 - (x_0 \cdot \hat{y}) \hat{y}}{r} \right| \\ &= \frac{r_0}{r} |\sin \delta_0|. \end{aligned}$$

Hence

$$A_0 = \left(1 - \frac{\sigma}{t} - \frac{r_0}{t}\cos\delta_0\right)|u_\rho|^2 \tag{4.8}$$
$$+ \left(1 + \frac{\sigma}{t} + \frac{r_0}{t}\cos\delta_0\right)\left(\frac{1}{2}|\Omega u|^2 + \frac{1}{4}|u|^4\right) + \frac{r_0}{t}\sqrt{2}|\sin\delta|u_\rho u_\alpha.$$

Estimating $|\Omega u| \geq |u_\alpha|$ as before, we have

$$A_0 \geq \left(2 - \frac{t_0 - r_0}{t}\right)|u_\rho|^2 - \frac{r_0}{2t}\left(\sqrt{2}\sqrt{1+\cos\delta_0}u_\rho - \sqrt{1-\cos\delta_0}u_\alpha\right)^2$$
$$+ \frac{t_0}{2t}\left(1 + \frac{r_0}{t_0}\right)|u_\alpha|^2 + \frac{t_0}{4t}\left(1 + \frac{r_0}{t_0}\cos\delta_0\right)|u|^4. \tag{4.9}$$

Note that all the latter terms are nonnegative for $z \in M(z_0), z_0 \in K$. Since $r_0 \leq |t_0|$ in (4.9), for $t \leq 2t_0$ we have $A_0 \geq |u_\rho|^2$. Moreover, given, $0 < \epsilon < 1$,$z_0 \in K$, let $t_m \leq 2t_0 < t_{m+1}$ and set

$$\Gamma = \Gamma(\epsilon : z_0) = \{z \in M_{t_m}(z_0) : |\delta_0| \leq \epsilon^{1/4}\}$$
$$\Delta = \Delta(\epsilon : z_0) = M_{t_m}(z_0)\backslash\Gamma.$$

Note that by (4.8) on Γ we have an estimate

$$A_0 \geq |u_\rho|^2 - \sqrt{2}\epsilon^{1/4}|u_\rho u_\alpha|$$
$$\geq |u_\rho|^2 - \sqrt{2}\epsilon^{1/4}d_{z_0}(u)$$

while, by (4.9), on Δ we have

$$A_0 \geq \frac{t_0}{4t}(1 + \frac{r_0}{t_0}\cos\delta_0)|u|^4$$
$$\geq \frac{1}{32}(1 - (1 - \frac{\epsilon^{1/2}}{2} + \epsilon))|u|^4$$
$$\geq \frac{\epsilon^{1/2}}{32}|u|^4 - \epsilon d_{z_0}(u).$$

Combining (4.7) and Lemma 4.1, we thus obtain

$$\int_\Gamma |u_\rho|^2\, do \leq \int_{M_{t_k}(z_0)} A_0\, do + \sqrt{2}\epsilon^{1/4}E_0 \tag{4.10}$$
$$\leq (\epsilon + \sqrt{2}\epsilon^{1/4})E_0 + o(1)\,(1 + log(1 + C(\epsilon))),$$

$$\begin{aligned}\frac{\epsilon^{1/2}}{32}\int_\Delta |u|^4\,do &\le \int_{M_{t_k}(z_0)} A_0\,do + \epsilon E_0\\ &\le 2\epsilon E_0 + o(1)\,(1+log(1+C(\epsilon)))\,,\end{aligned} \tag{4.11}$$

$$\begin{aligned}\int_{M_{t_l}^{t_m}(z_0)} |u_\rho|^2\,do &\le \int_{M_{t_k}^{t_m}(z_0)} A_0\,do + \int_{M_{t_k}^{t_m}(z_0)} |u_\rho|^2\,do\\ &\le 3\epsilon E_0 + o(1)log(1+C(\epsilon)),\end{aligned} \tag{4.12}$$

where $o(1)\to 0$ as $l\to\infty$, we may assume that $t_l \le t_k \le t_m$.

Proof of Theorem 0.1. Given $\epsilon > 0$, we split the integral in (1.9) and use Hölder's inequality as follows

$$\begin{aligned}&\int_{M_{t_l}} \frac{|u|^2}{|z-z_0|^2}\,do\\ &\le \int_\Gamma \frac{|u|^2}{|z-z_0|^2}\,do + \int_\Delta \frac{|u|^2}{|z-z_0|^2}\,do + \int_{M_{t_l}^{t_m}} \frac{|u|^2}{|z-z_0|^2}\,do.\end{aligned}$$

By Lemma 2.1 and (4.10)

$$\begin{aligned}&\int_\Gamma \frac{|u|^2}{|z-z_0|^2}\,do\\ &\le\frac{4}{3}\int_\Gamma |u_\rho|^2\,do + \frac{1}{6}|t_m - t_0|^{-1}\int_{\partial D(t_m:z_0)} |u|^2\,do\\ &\le\frac{4}{3}(\epsilon+\sqrt{2}\epsilon^{1/4})E_0 + o(1)\,(1+log(1+C(\epsilon))) + C\left(\int_{\partial D(t_m:z_0)} |u|^3\,do\right)^{2/3}\end{aligned}$$

By Lemma 2.1 and Lemma 3.2

$$\begin{aligned}&\int_{\partial D(t_m:z_0)} |u|^3\,do\\ &\le C\left\{\left(\int_{D(t_m)} |u|^4\,dx\right)^{1/2}\left(\int_{D(t_m)} |\nabla u|^2\,dx\right)^{1/2} + \left(\int_{D(t_m)} |u|^4\,dx\right)^{3/4}\right\}\\ &\le C\left\{\left(\int_{D(t_m)} |\nabla u|^2\,dx\right)^{1/2} + \left(\int_{D(t_m)} |u|^4\,dx\right)^{3/2}\right\}\left(\int_{D(t_m)} |u|^4\,dx\right)^{1/2}\\ &\le C(E_0)o(1),\end{aligned}$$

where $o(1) \to 0$ as $m \geq l$ tend to infinity. Similarly, by Lemma 2.1, Lemma 3.2 and (4.12)

$$\begin{aligned}
&\int_{M_{t_l}^{t_m}} \frac{|u|^2}{|z-z_0|^2}\, do \\
\leq &\frac{4}{3}\int_{M_{t_l}^{t_m}(z_0)} |u_\rho|^2\, do + \frac{1}{6}|t_m - t_0|^{-1}\int_{\partial D(t_l:z_0)} |u|^2\, do \\
\leq &4\epsilon E_0 + o(1)log(1+C(\epsilon)) + o(1)C(E_0).
\end{aligned}$$

Finally, by (4.11),

$$\begin{aligned}
&\int_\Delta \frac{|u|^2}{|z-z_0|^2}\, do \\
\leq &\left(\int_\Delta \frac{|u|^{4/3}}{|z-z_0|^{8/3}}\, do\right)^{3/4}\left(\int_\Delta |u|^4\, do\right)^{1/4} \\
= &64\epsilon^{1/2}\left(\int_\Delta \frac{|u|^{4/3}}{|z-z_0|^{8/3}}\, do\right)^{3/4} E_0 + o(1)\epsilon^{1/2}\left(1 + log(1+C(\epsilon))\right).
\end{aligned}$$

Hence, if we first choose $\epsilon > 0$ sufficiently small and then choose $l \in N$ sufficiently large, then the integral

$$\int_{M_{t_l}(z_0)} \frac{|u|^2}{|z-z_0|^2}\, do$$

can be made as small as we please. //////

REMARK. While preparing this paper, we were informed that M.G. Grillakis has obtained the similar result. Our proof is independent from his proof and is based on a different view point.

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