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A GENERALIZATION OF THE HARADA THEOREM

SANG CHEOL LEE

Throughout this note, we shall assume that every ring R (not necessarily commutative) has an identity and every module is a unitary left module.

The Harada theorem says that if A is a direct summand of a direct sum of indecomposable injective modules and if A is a nonsingular module, then A itself is a direct sum of completely indecomposable injective modules. This paper proves that *every nonsingular homomorphic image of a sum of indecomposable injective submodules of a module is a direct sum of indecomposable injective modules.* Further, note that every indecomposable injective module is completely indecomposable injective. This provides us with the natural generalization of the theorem and consequently a new proof is given.

Let $\{ M_i \}_{i \in I}$ be a family of submodules of an R-module M. Let N be any homomorphic image of $\sum_{i \in I} M_i$. Then there is a homomorphism f from $\sum_{i \in I} M_i$ onto N. If we put $N_i = f(M_i)$ for all $i \in I$, then $N = \sum_{i \in I} N_i$.

We can now consider the family $\{ N_i \}_{i \in I}$. By Zorn's lemma, there is a maximal collection $\mathcal{C}$ of members of $\{ N_i \}_{i \in I}$ such that $\sum_{N' \in \mathcal{C}} N'$ is a direct sum.

Let E be any indecomposable injective R-module and let E' be any non-zero submodule of E. Then E is an injective hull for E' [SV 72, Prop.2.28] and hence E/E' is singular [GW 89, Prop.3.26]. If E/E' is nonsingular, then $E/E' = 0$ and hence $E' = E$. Therefore the only nonsingular homomorphic images of an indecomposable injective R-module are zero and an indecomposable injective R-module (which is isomorphic to E).

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Assume further that each M_i is indecomposable injective and that N is nonsingular. It follows from the above argument that we may assume that each N_i is indecompsable injective.

It is fairly well known that for any prime p in the ring $\mathbb{Z}$ of integers, the $\mathbb{Z}$-module $G = \mathbb{Z}(p^\infty) \oplus \mathbb{Z}(p^\infty)$ has the property that not every submodule has a unique injective hull in G. However, it is also well known that every submodule has a unique injective hull, within a given nonsingular R-module. (In fact, this follows from Prop.s 4.9, 3.28(b), 3.26 in [G89], and Lem. 2.1 in [G 76].)

We claim that $N = \sum_{N' \in \mathcal{C}} \oplus N'$. In fact, put $N^* = \sum_{N' \in \mathcal{C}} \oplus N'$ and suppose on the contrary that $N \neq N^*$. Then one, say N_k, of the N_i's is not contained in N^*. By the maximality of $\mathcal{C}$, we have $N_k \cap N^* \neq 0$. We can now pick out a finite collection $N_1, \cdots, N_r$ of members of $\mathcal{C}$ such that

$$N_k \cap (N_1 \oplus \cdots \oplus N_r) \neq 0.$$

Since $N_1 \oplus \cdots \oplus N_r$ is injective, $N_k \cap (N_1 \oplus \cdots \oplus N_r)$ has an injective hull which is a submodule of $N_1 \oplus \cdots \oplus N_r$ [SV 72, Prop.2.22]. Further, N_k is an injective hull for $N_k \cap (N_1 \oplus \cdots \oplus N_r)$. Therefore, by the uniqueness,

$$N_k \subseteq N_1 \oplus \cdots \oplus N_r \subseteq N^*,$$

which contradicts.

Let us summarize the results.

THEOREM. *Let $\{ M_i \}_{i \in I}$ be a family of indecomposable injective submodules of an R-module M. Then every nonsingular homomorphic image of $\sum_{i \in I} M_i$ is a direct sum of indecomposable injective R-modules.*

COROLLARY 1. *Suppose, in addition to the hypothesis of the theorem, that A is a direct summand of $\sum_{i \in I} \oplus M_i$.*

(1) *[H83, (8.2.7)] If A is a nonsingular R-module, then A is a direct sum of indecomposable injective R-modules.*

(2) *If $A/Z(A)$ is a nonsingular R-module, then $A/Z(A)$ is a direct sum of indecomposable injective R-modules, where $Z(A)$ denotes the (maximal) singular submodule of A.*

COROLLARY 2. *Suppose, in addition to the hypothesis of the theorem, that A is a direct summand of $\sum_{i\in I} \oplus M_i$. If R is a nonsingular ring, then $A/Z(A)$ is a direct sum of indecomposable injective R-modules.*

References

[G76] Goodearl, K. R., *Ring Theory (Nonsingular Rings and Modules)*, Marcel Dekker, Inc., 1976.

[GW89] Goodearl, K. R. and Warfield, R. B., Jr., *An Introduction to Noncommutative Noetherian Rings*, Cambridge University Press, 1989.

[H83] Harada, M., *Factor Categories with Applications to Direct Decomposition of Modules*, Marcel Dekker, Inc., 1983.

[SV72] Sharpe, D. W. and Vámos, P., *Injective Modules*, Cambridge University Press, 1972.

Department of Mathematics Education
Chonbuk National University
Chonju, Chonbuk 560-756, Korea