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FIBRED RIEMANNIAN SPACES WITH CRITICAL RIEMANNIAN METRICS*

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1. Introduction

Let M be a compact orientable Riemannian manifold and let $\mu(M)$ be the space of C^∞ Riemannian metrics G on M satisfying $\int_M dV_G = 1$, where dV_G is the volume element measured by G. For an element G in $\mu(M)$, we assume that $f(\kappa)$ is a scalar field on M determined by G as the contraction of a tensor product of the curvature tensor. Then $H_M[G] = \int_M f(\kappa) dV_G$ defines a mapping $H_M : \mu(M) \longrightarrow R$. From now on, we denote H_M by H. In this case, a critical point of H is called a critical Riemannian metric with respect to the field $f(\kappa)$ and denoted by G_H (cf. [1.6]).

Following M. Berger [1], we have four kinds of critical Riemannian metrics G_A, G_B, G_C and G_D as the most prominent ones. The corresponding integrals are

$$A_M[G] = \int_M K dV_G, \qquad B_M[G] = \int_M K^2 dV_G,$$

$$C_M[G] = \int_M S^2 dV_G, \qquad D_M[G] = \int_M R^2 dV_G,$$

where R, S and K are the Riemannian curvature tensor, Ricci curvature tensor and scalar curvature respectively. The equations of the critical Riemannian metrics obtained by M. Berger can be written in the following form in tensor notations

$$\begin{aligned} A_{ji} &= c_A G_{ji}, \qquad B_{ji} = c_B G_{ji}, \\ C_{ji} &= c_C G_{ji}, \qquad D_{ji} = c_D G_{ji}, \end{aligned} \tag{1.1}$$

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where c_A, c_B, c_C and c_D are undetermined constants and A_{ji}, B_{ji}, C_{ji} and D_{ji} are given by

$$A_{ji} = -S_{ji} + \frac{1}{2}KG_{ji}, \tag{1.2}$$

$$B_{ji} = 2\nabla_j\nabla_i K - 2(\nabla_k\nabla^k K)G_{ji} - 2KS_{ji} + \frac{1}{2}K^2G_{ji}, \tag{1.3}$$

$$\begin{aligned} C_{ji} = \nabla_j\nabla_i K - \nabla_k\nabla^k S_{ji} - \frac{1}{2}(\nabla_k\nabla^k K)G_{ji} \\ - 2R_{jkhi}S^{kh} + \frac{1}{2}S_{kh}S^{kh}G_{ji}, \end{aligned} \tag{1.4}$$

$$\begin{aligned} D_{ji} = 2\nabla_j\nabla_i K - 4\nabla_k\nabla^k S_{ji} + 4S_{jk}S_i{}^k - 4R_{jkhi}S^{kh} \\ - 2R_{jkhl}R_i{}^{khl} + \frac{1}{2}R_{khlm}R^{khlm}G_{ji}, \end{aligned} \tag{1.5}$$

where ∇ is the Riemannian connection with respect to G on M.

The purpose of this paper is to study the fibred Sasakian space forms with critical Riemannian metric and we get, in this case, M is a space of constant curvature, the base space is a complex space form and each fibre is 1-dimension and totally geodesic. Essential examlpes for this result can be found in [4].

2. Fibred Riemannian spaces

Let $\{M,\ N,\ G,\ \pi\}$ be a fibred Riemannian space, that is, $\{M,\ G\}$ is an m-dimensional total space with Riemannian metric G, N an n-dimensional base space, and $\pi : M \longrightarrow N$ the projection with maximum rank n. The fibre passing through a point $P \in M$ is denoted by $\overline{M}(P)$ or generally $\overline{M}$, and the metric tensor G is projectable. Throughout this paper, the range of indices are as follows;

$$h,\ i,\ j,\ k,\ l = 1,\ 2, \ldots,\ m,$$

$$a,\ b,\ c,\ d,\ e = 1,\ 2, \dots,\ n,$$

$$x,\ y,\ z,\ u,\ v = n+1, \dots,\ n+p = m.$$

One of the present author [5] proved that if the Sasakian structure (ϕ, ξ, η) on M is of constant ϕ-holomorphic sectional curvature k, that is, the curvature tensor R of M is given by

$$\begin{aligned} R_{kji}{}^{h} = {} & \frac{k+3}{4}(G_{ji}\delta_k{}^h - G_{ki}\delta_j{}^h) \\ & - \frac{k-1}{4}(\eta_j\eta_i\delta_k{}^h - \eta_k\eta_i\delta_j{}^h + G_{ji}\eta_k\xi^h \\ & - G_{ki}\eta_j\xi^h - \phi_{ji}\phi_k{}^h + \phi_{ki}\phi_j{}^h + 2\phi_{kj}\phi_i{}^h), \end{aligned} \tag{2.1}$$

then the base space is a complex space form, each fibre is minimal and

$$\begin{aligned} \overline{R}_{uzy}{}^{x} = {} & \frac{k+3}{4}(\overline{g}_{zy}\delta_u{}^x - \overline{g}_{uy}\delta_z{}^x) \\ & - \frac{k-1}{4}(\overline{\eta}_z\overline{\eta}_y\delta_u{}^x - \overline{\eta}_u\overline{\eta}_y\delta_z{}^x + \overline{g}_{zy}\overline{\eta}_u\overline{\xi}^x \\ & - \overline{g}_{uy}\overline{\eta}_z\overline{\xi}^x - \overline{\phi}_{zy}\overline{\phi}_u{}^x + \overline{\phi}_{uy}\overline{\phi}_z{}^x + 2\overline{\phi}_{uz}\overline{\phi}_y{}^x) \\ & + h_{zye}h_u{}^{xe} - h_{uye}h_z{}^{xe}, \end{aligned} \tag{2.2}$$

$$S_{zy} = \overline{S}_{zy} + \nabla^*{}_e h_{zy}{}^e + n\eta_z\eta_y, \tag{2.3}$$

$$L_{cb}{}^x = J_{cb}\overline{\xi}^x, \tag{2.4}$$

where we have put

$$\nabla^*{}_d h_{xy}{}^a = \partial_d h_{xy}{}^a + \Gamma_d{}^a{}_e h_{xy}{}^e - Q_{dx}{}^z h_{zy}{}^a - Q_{dy}{}^z h_{xz}{}^a, \tag{2.5}$$

$$Q_{dx}{}^y = P_{dx}{}^y - h_x{}^y{}_d, \tag{2.6}$$

$P_{dx}{}^y$ are local functions related to $\mathcal{L}_{C_x}C^y = P_{dx}{}^y E^d$, $\{E^a, C^x\}$ are dual to the local frame $\{E_b, C_y\}$, J is the complex structure induced on N, $(\overline{\phi}, \overline{\xi}, \overline{\eta}, \overline{g})$ is the Sasakian structure induced on $\overline{M}$, $S_{xy} = S(C_x, C_y)$, $\overline{R}$ and $\overline{S}$ are the curvature tensor and Ricci curvature tensor on $\overline{M}$ respectively, $h = (h_{xy}{}^a)$ and $L = (L_{cb}{}^x)$ are the components of the second fundamental tensor and normal connection of each fibre $\overline{M}$ respectively.

3. Main results

First we show that, if the Riemannian metric G on the Sasakian space form M is a critical Riemannian metric G_A, then each fibre is totally geodesic and the dimension of $\overline{M}$ is one.

From (2.1), we get

$$S_{ji} = \frac{(m+1)k+3m-5}{4}G_{ji} - \frac{(m+1)(k-1)}{4}\eta_j\eta_i \tag{3.1}$$

and since M is an Einstein space, it is easily seen that $k = 1$ by use of the fact that $m \neq 1$. Therefore M becomes a space of constant curvature and

$$\| h_{xy}{}^a \|^2 = -n(p-1)$$

by use of (2.2), (2.3) and (3.1). Since $\| h_{xy}{}^a \|^2$ is non-negative, we have $p = 1$ and that h vanishes identically. Thus we have

THEOREM 3.1. *Let M be the fibred Sasakian space with constant ϕ-holomorphic sectional curvature k. If G is a critical Riemannian metric G_A, then M has 1-dimensional totally geodesic fibres, M is a space of constant curvature and the base space N is a complex space form and the Riemannian metric g on N is a critical Riemannian metric G_A.*

By the same argument of Theorem 3.1, we get

THEOREM 3.2. *Let M be the fibred Sasakian space with constant ϕ-holomorphic sectional curvature k. If G is a critical Riemannian metric G_B, then we have the same result of Theorem 3.1 except that g on N is a critical Riemannian metric G_B.*

Assume that G is a critical Riemannian metric G_C, then

$$\nabla_k\nabla^k S_{ji} = -\frac{(m+1)(k-1)}{2}(G_{ji} - \eta_j\eta_i), \tag{3.2}$$

$$\begin{aligned} R_{jkhi}S^{kh} = {} & \{(m-1)\alpha\gamma + 2\beta\gamma - \alpha\delta + \beta\delta\}G_{ji} \\ & + \{\alpha\delta - \beta\delta - (m+1)\beta\gamma\}\eta_j\eta_i\ , \end{aligned} \tag{3.3}$$

$$S_{ji}S^{ji} = m\alpha^2 - 2\alpha\beta + \beta^2, \tag{3.4}$$

where we have put

$$\alpha = \frac{k+3}{4}, \qquad \beta = \frac{k-1}{4},$$

$$\gamma = \frac{(m+1)k+3m-5}{4}, \qquad \delta = \frac{(m+1)(k-1)}{4}.$$

From (1.1), (1.4) and (3.2)~(3.4), we see that the coefficient of $\eta_j\eta_i$ in $C_{ji} - c_C G_{ji}$ is $\frac{1}{8}(m+1)(k-1)\{(m+1)k-m-9\}$. Hence $k=1$ or $k = \frac{m+9}{m+1}$.

(I) if $k=1$, then M is a space of constant curvature and $\| h_{xy}{}^a \|^2 = -n(p-1)$ by use of (2.2) and (2.3), so $p=1$ and $h=0$.

(II) if $k = \frac{m+9}{m+1}$, then

$$\| h_{xy}{}^a \|^2 = \frac{p-1}{m+1}\{(m+3)p+2\} - (m+2)(p-1),$$

that is,

$$(m+1)\| h_{xy}{}^a \|^2 = (1-p)(m+3)(m-p). \tag{3.5}$$

Since $(m+1)\| h_{xy}{}^a \|^2 \geq 0$, we get $p \leq 1$. Hence we have $p=1$ and that $h=0$. Thus we obtain

THEOREM 3.3. *If the fibred Sasakian space M with constant ϕ-holomorphic sectional curvature k has a critical Riemannian metric G_C, then*

(1) M is a space of constant curvature,

(2) M has 1-dimensional totally geodesic fibres,

(3) the base space N is a complex space form and g on N is a critical Riemannian metric G_C.

Finally, if the total space M has a critical Riemannian metric G_D, then

$$S_{jt}S_i{}^t = \gamma^2 G_{ji} + \delta(\delta - 2\gamma)\eta_j\eta_i\ , \tag{3.6}$$

$$\begin{aligned} R_{jlkh}R_i{}^{lkh} = {} & \{\frac{1}{8}(k+3)(m-1) + (k+3)(k-1)\}G_{ji} \\ & - \frac{1}{2}(k+3)(k-1)(m+1)\eta_j\eta_i\ . \end{aligned} \tag{3,7}$$

Since the coefficient of $\eta_j\eta_i$ in $D_{ji} - c_D G_{ji}$ vanishes identically, we have

$$(k-1)\{k(m+1)-(6m^2+3m-11)\}=0, \tag{3.8}$$

that is, $k=1$ or $k=\frac{6m^2+3m-11}{m+1}$.

For the case of $k=1$, we have same result of (I) by the similar argument.

If $k=\frac{6m^2+3m-11}{m+1}$, then we obtain

$$S_{ji} = \frac{3m^2+3m-8}{2}G_{ji} - \frac{3m^2+m-6}{2}\eta_j\eta_i\ , \tag{3.9}$$

$$K = \frac{3(m-1)^2(m+2)}{2} \tag{3.10}$$

and that

$$2(m+1)\| \; h_{xy}{}^a \; \|^2 = (1-p)(m-p)(3m^2+3m-4) \tag{3.11}$$

by use of (2.2), (2.3), (3.9) and (3.10).

Since $3m^2+3m-4>0$, we get $p=1$ and that $h=0$. Thus we have

THEOREM 3.4. *If the fibred Sasakian space M with constant ϕ-holomorphic sectional curvature k has a critical Riemannian metric G_D, then*

(1) M is a space of constant curvature,

(2) M has 1-dimensional totally geodesic fibres,

(3) the base space N is a complex space form and g on N is a critical Riemannian metric G_D.

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