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An $O(h^6)$ Quintic Spline Interpolation for Quintic Spline Collocation Method

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ABSTRACT. An quintic spline interpolate to a function in $C^{10}[a,b]$ and its $O(h^6)$ error behavior are presented when its fourth derivative satisfies some kind of end conditions. The $O(h^6)$ relations between its derivatives up to fourth order and the m-th derivatives of the given function are also given at the nodes.

1. Introduction

This paper is a continuation of [1], [2] and [3]. In [2], an $O(h^6)$ quintic spline collocation method were developed and analyzed for fourth order linear two-point value problem

$$\text{(1-1)} \qquad \mathbf{L}u \equiv D^4u(x) + \sum_{m=0}^{3} a_k(x)D^m u(x) = f(x), \quad x \in (a,b)$$

with the boundary condition

$$\text{(1-2)} \qquad \mathbf{B}u \equiv \sum_{m=0}^{3} \alpha_{im}D^m u(a) + \beta_{im}D^m u(b) = g_i, \quad i = 0,1,2,3.$$

The optimal method in [2] finds a quintic spline approximation to the solution by forcing it to satisfy a perturbation of the given original operator at the nodes and the auxilary end conditions, see section 3 in [2]. But it is based on Theorem 2.1 in [2], which is not correct. In fact

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the $O(h^6)$ error estimates of Theorem 2.1 in [2] is proved under the assumption that there exists the unique quintic spline interpolate to a given u satisfying 12 auxilary end conditions, which is not true and hence the method itself may not be correct. We will show here that the method is correct by presenting an $O(h^6)$ quintic spline interpolate to a given function $u \in C^{10}[a, b]$ which has the same approximation property that the interpolate in [2] does. The same error was also made in [1] and [3]. An $O(h^6)$ quintic spline collocation method were developed for second order two-point boundary value problem in [1]. But noticing that the error may be corrected in the same way, we will confine ourself only to [2] in section 2 for convenience.

2. Quintic spline interpolation

Given $n \geq 1$, let $Q(\Pi) \equiv \{u \in C^4[a, b] \mid u|_{[x_k, x_{k+1}]} \in P_5,\ k = 0, 1, \ldots, n-1\}$, where P_5 is the set of all polynomials of degree ≤ 5 and Π the uniform partition of the interval $[a, b]$

$$\Pi : x_0 < x_1 < \cdots < x_n,\ x_k = a + kh,\ k = 0, 1, \ldots, n,\ h = (b-a)/n.$$

Then we may represent any quintic spline $S \in Q(\Pi)$ by

$$S(x) = \sum_{i=-2}^{n+2} c_i B_i(x),$$

where B_i, for $-2 \leq i \leq n+2$, is the quintic B-spline with the support $[x_{i-3}, x_{i+3}]$ over the extended partition

$$x_{-5} < x_{-4} < \cdots < x_{n+4} < x_{n+5},\ x_k = a + kh,$$
$$-5 \leq k \leq n+5,\ h = (b-a)/n.$$

From now on we denote by $u_k \equiv u(x_k)$ and $u_k^{(m)} \equiv D^m u(x_k)$ for all integer $m \geq 1$ and for any given function u.

LEMMA 1. *For a given function $u \in C^{10}[a,b]$, there exists the unique quintic spline interpolate*

$$S = \sum_{i=-2}^{n+2} c_i B_i(x) \in Q(\Pi)$$

to u satisfying the interpolation conditions:

$$S_k = u_k, \quad k = 0, 1, \ldots, n \tag{2-1a}$$

and the auxilary end conditions:

$$S_k^{(4)} = u_k^{(4)} - \frac{h^2}{12} u_k^{(6)} + \frac{h^4}{240} u_k^{(8)}, \; k = 0, 1, n-1, n. \tag{2-1b}$$

PROOF. By using the values of $B_i(x)$ and $B_i^{(4)}(x)$, $-2 \le i \le n+2$, at the nodes:

x	x_{i-3}	x_{i-2}	x_{i-1}	x_i	x_{i+1}	x_{i+2}	x_{i+3}
$B_i(x)$	0	1	26	66	26	1	0
$B_i^{(4)}(x)$	0	$\frac{120}{h^4}$	$\frac{-480}{h^4}$	0	$\frac{-480}{h^4}$	$\frac{120}{h^4}$	0

it is easy to construct a linear system with the unknowns c_i, $-2 \le i \le n+2$, whose coefficient matrix is row-equivalent to a strictly diagonally domonant matrix and hence that has a unique solution.

The linear dependence relations connecting a quintic spline S on the uniform partition Π and its derivatives in Lemma 2 will be used as a basis to prove Theorem 1, which in turn gives us a correct basis to construct the $O(h^6)$ quintic spline collocation method of [2].

LEMMA 2. *Let S be a quintic spline on the uniform partition Π. Then the following recurrence relations hold*

(2-2)

$$h^2 S_k^{(2)} = (S_{k-1} - 2S_k + S_{k+1}) - \frac{h^4}{120}(S_{k-1}^{(4)} + 8S_k^{(4)} + S_{k+1}^{(4)}),\ k = 1,\dots,n-1,$$

(2-3)

$$S_{k-1}^{(2)} - 2S_k^{(2)} + S_{k+1}^{(2)} = \frac{h^2}{6}(S_{k-1}^{(4)} + 4S_k^{(4)} + S_{k+1}^{(4)}),\ k = 1,\dots,n-1,$$

(2-4)

$$h^3 S_k^{(3)} = (-S_{k-1} + 3S_k - 3S_{k+1} + S_{k+2}) + \frac{h^4}{120}(S_{k-1}^{(4)} - 33S_k^{(4)} - 27S_{k+1}^{(4)} - S_{k+2}^{(4)}),\ k = 1,\dots,n-2,$$

(2-5)

$$h^3 S_k^{(3)} = (-S_{k-2} + 3S_{k-1} - 3S_k + S_{k+1}) + \frac{h^4}{120}(S_{k-2}^{(4)} + 27S_{k-1}^{(4)} + 33S_k^{(4)} - S_{k+1}^{(4)}),\ k = 2,\dots,n-1,$$

(2-6)

$$-S_k^{(3)} + S_{k+1}^{(3)} = \frac{h}{2}(S_k^{(4)} + S_{k+1}^{(4)}),\ k = 0,\dots,n-1,$$

(2-7)

$$hS_k^{(1)} = (S_{k+1} - S_k) - \frac{h^2}{20}(7S_k^{(2)} + 3S_{k+1}^{(2)}) + \frac{h^3}{60}(-3S_k^{(3)} + 2S_{k+1}^{(3)}),\ k = 0,\dots,n-1,$$

(2-8)

$$hS_k^{(1)} = (S_k - S_{k-1}) + \frac{h^2}{20}(3S_{k-1}^{(2)} + 7S_k^{(2)}) + \frac{h^3}{60}(2S_{k-1}^{(3)} - 3S_k^{(3)}),\ k = 1,\dots,n.$$

PROOF. The relation (2-3) is Lemma 1 in [4] with $N = 5, p = 4, q = 2$ and the relation (2-2) follows if you use Theorem 1 in [4] together with (2-3). The rest of the relations can be verified by using the results of [4]. See Theorem 1, Theorem 2 and Lemma 1 in [4]. □

THEOREM 1. *Let $S \in Q(\Pi)$ be the unique quintic spline interpolate to a given function $u \in C^{10}[a, b]$ satisfying the conditions (2-1). Then we have for $k = 0, 1, \ldots, n-1, n$*

$$S_k^{(1)} = u_k^{(1)} + O(h^6), \tag{2-9a}$$

$$S_k^{(2)} = u_k^{(2)} + \frac{h^4}{720} u_k^{(6)} + O(h^6), \tag{2-9b}$$

$$S_k^{(3)} = u_k^{(3)} - \frac{h^4}{240} u_k^{(7)} + O(h^6), \tag{2-9c}$$

$$S_k^{(4)} = u_k^{(4)} - \frac{h^2}{12} u_k^{(6)} + \frac{h^4}{240} u_k^{(8)} + O(h^6), \tag{2-9d}$$

and the error estimates (2-10) holds

$$\|(u - S)^{(m)}\|_\infty = O(h^{(6-m)}), \; m = 0, 1, 2, 3, 4. \tag{2-10}$$

PROOF. Refer to Theorem 2.1 in [2] for the proof of (2-9d) and (2-10). After inserting the relation (2-9d) into (2-4) and (2-5), using the fact S is the interpolate to u and expanding them in Taylor series, we can show that the result (2-9c) holds for $k = 1, \ldots, n-1$. To get the approximations of $S_0^{(3)}$ and $S_n^{(3)}$, we use the relation (2-6) with $k = 0, n-1$ by the same manner together with (2-9c) for $k = 1, n-1$ and (2-9d) for $k = 0, 1, n-1, n$ repectively. To prove (2-9b), $k = 1, \ldots, n-1$ and (2-9b), $k = 0, n$, we use, by the same manner, the relations (2-2) and (2-3) respectively together with (2-1) and (2-9d). We can also prove (2-9a) by the same argument using the relations (2-1), (2-7), (2-8), (2-9b) and (2-9c).

In Theorem 2.1 in [2], the same approximation results as in the above Theorem 1 was proved assuming that there exists the unique quintic interpolate $S \in Q(\Pi)$ satisfying (2-1a), (2-1b) and 8 more auxilary end conditions

$$S_k^{(2)} = u_k^{(2)} + \frac{h^4}{720} u_k^{(6)}, \; k = 0, 1, n-1, n, \tag{2-11}$$

$$S_k^{(3)} = u_k^{(3)} - \frac{h^4}{240} u_k^{(7)}, \; k = 0, 1, n-1, n, \tag{2-12}$$

by using the relations which are different from those in Lemma 2. But the existence of such a quintic spline interpolate does not hold since the dimension of $Q(\Pi)$ is only $n+5$ and the number of conditions is $n+13$ instead of $n+5$. Therefore the approximations in Theorem 2.1 of [2], and hence the $O(h^6)$ quintic spline collocation method in [2] based on them, may not be correct. But the method fortunately turns out to be the case by Theorem 1.

REFERENCES

1. M.Irodotou-Ellina, E.N.Houstis and S.B. Kim, *An $O(h^6)$ quintic spline collocation methods for second order two-point boundary value problems*, Purdue University, CSD-TR-904, 1991.
2. M.Irodotou-Ellina and E.N.Houstis, *An $O(h^6)$ quintic spline collocation methods for fourth order two-point boundary value problems*, BIT **28** (1988), 288–301.
3. M.Irodotou-Ellina and E.N.Houstis, *An $O(h^6)$ quintic spline collocation methods for fourth order two-point boundary value problems*, Purdue University, CSD-TR-618, 1986.
4. D.J. Fyfe, *Linear dependence relations connecting equal interval Nth degree splines and their derivatives*, J. Inst. Maths Applics **7** (1971), 398–406.

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