JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume 7, June 1994

Upper-Lower Solutions for Singular Elliptic Equations under Nonlinear Boundary Conditions

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1. Introduction

Our interest is to study the upper-lower solution method for degenerate elliptic equations under the nonlinear Robin-type boundary condition. The method of upper-lower solutions is well known for uniformly elliptic operators under the usual boundary condition. (See [6], [2].)

In this paper, we will prove the upper-lower solution method for singular elliptic equations under the nonlinear boundary conditions. The technique employs Schauder's fixed point theorem, which is a new justification of the upper-lower solution method.

2. Results

Throughout this paper, let $\mathbf{K} = C(\bar{\Omega})^+$ be the positive cone of the ordered Banach space $C(\bar{\Omega})$ where Ω is a bounded region in $\mathbf{R}^n$ and denote the ordered interval $[[u_1, u_2]] := \{u \in C(\bar{\Omega}) : u_1 \leq u \leq u_2 \quad \text{for } u_1,\, u_2 \in C(\bar{\Omega})\}$.

Let G be Green's function: $-\Delta G = \delta(x, y)$. We denote Green's functions under the Dirichlet and the Robin condition $\cdot + \lambda \frac{\partial \cdot}{\partial n} = 0$, $\lambda > 0$, by G_D and $G_{R,\lambda}$, respectively. Then we have the following lemma.

LEMMA 1.
(i) $\frac{\partial G_D}{\partial n} < 0$ *and* $\frac{\partial G_{R,\lambda}}{\partial n} < 0$

Received by the editors on June 30, 1994.
1980 *Mathematics subject classifications*: Primary 35J60.

(ii) $\|G_{R,\lambda} - G_D\|_{C^{2,\alpha}(\bar{\Omega})} = O(\lambda)$, $\|G_{R,\lambda} - G_D\|_{W^{1,p}(\partial\Omega)} = O(\lambda)$, *for an arbitrary* $p > 0$, $\alpha \in (0,1)$, *as* $\lambda \to 0^+$.

PROOF. (i) See [5].
(ii) First note $-\Delta(G_{R,\lambda} - G_D) = \delta - \delta = 0$ in Ω and $(G_{R,\lambda} - G_D) + \lambda(\frac{\partial(G_{R,\lambda} - G_D)}{\partial n}) = G_{R,\lambda} + \lambda\frac{\partial G_{R,\lambda}}{\partial n} - \lambda\frac{\partial G_D}{\partial n} = -\lambda\frac{\partial G_D}{\partial n} > 0$ on $\partial\Omega$ by (i). Let $x_0 \in \bar{\Omega}$ be such that $(G_{R,\lambda} - G_D)(x_0) = \max_{\bar{\Omega}}(G_{R,\lambda} - G_D)$. The strong maximum principle implies that $x_0 \in \partial\Omega$ and therefore $G_{R,\lambda} - G_D > 0$ at x_0. It follows from Hopf's lemma that $\frac{\partial(G_{R,\lambda} - G_D)(x_0)}{\partial n} > 0$. Notice that $\frac{\partial G_D(x,y)}{\partial n}$ for $x \in \Omega$, $y \in \partial\Omega$ is a smooth function of $y \in \partial\Omega$. So we have $\|G_{R,\lambda} - G_D\|_{C(\bar{\Omega})} \leq \lambda \max_{x\in\partial\Omega} |\frac{\partial G_D(x)}{\partial n}| \to 0$ as $\lambda \to 0^+$. Let $G = G_{R,\lambda} - G_D$. Then $\|G\|_{C(\bar{\Omega})} = O(\lambda)$. By the elliptic regularity of the Robin problem (see sec.6.7 in [3]), $\|G\|_{C^{2,\alpha}(\bar{\Omega})} = O(\lambda)$, $\alpha \in (0,1)$. In particular, by the trace mapping theorem (see sec.8 in [7]), we have $\|G_{R,\lambda} - G_D\|_{W^{1,p}(\partial\Omega)} = O(\lambda)$ as $\lambda \to 0^+$.

We define the class $G \subset C(\bar{\Omega} \times \mathbf{R}^+)$ as follows: Let $\varphi = \varphi(x,\xi) \in C(\bar{\Omega} \times \mathbf{R})$ and φ be C^1-function in ξ. We say that $\varphi \in G$ if and only if $\varphi(x,\xi) \geq \varphi_0(x) \geq 0$ for $x \in \Omega$, $\xi \in \mathbf{R}^+$ and $\frac{1}{\varphi_0} \in L^m(\Omega)$ where $m > n$, and φ is nondecreasing and concave down in $\xi \in \mathbf{R}^+$.

Here is the lemma which plays a primary role in this paper.

LEMMA 2. *Let* $P > 0$ *be a constant and assume* $\varphi \in G$. *Also let* $0 \not\equiv h \geq 0$, $h \in C(\bar{\Omega})$. *Consider*

$$(1) \qquad \begin{cases} -\varphi(x,u)\Delta u + Pu = h \\ \dfrac{\partial u}{\partial n} + \beta(u) = 0 \qquad \text{on } \partial\Omega \end{cases}$$

where $\beta \in C^2$, $\beta(0) = 0$, β *is strictly increasing.*

Then (1) has a unique positive solution $u \in W^{2,m}(\Omega) \bigcap C^{1,\alpha}(\bar{\Omega})$ *for some* $\alpha \in (0,1)$. *Moreover, the solution operator* S *such that* $u = Sh$ *is compact in* $\mathbf{K}$ *or* $C(\bar{\Omega})$, *where* $\mathbf{K}$ *is the positive cone of* $C(\bar{\Omega})$.

PROOF. The positiveness of solutions to (1) for $h \geq 0$ follows from the generalized maximum principle. (See page 209 in [3].)

We claim that the nonnegative solution to (1) when $0 \leq h \in C(\bar{\Omega})$ is unique. Let u and v be two nonnegative solutions of (1) such that $u \not\equiv v$. Without loss of generality, let $\min_{x\in\bar{\Omega}}(u(x)-v(x)) < 0$. (The other case is similar.) Assume $u(x_0)-v(x_0) = \min_{x\in\bar{\Omega}}(u(x)-v(x)) < 0$. One can show that $x_0 \notin \partial\Omega$. In fact, if $x_0 \in \partial\Omega$, then by the maximality at x_0, $\frac{\partial(u-v)(x_0)}{\partial n} \leq 0$. Also $\beta(u(x_0)) - \beta(v(x_0)) < 0$ because β is strictly increasing. Thus the boundary condition becomes $[\frac{\partial u(x_0)}{\partial n} - \frac{\partial v(x_0)}{\partial n}]+\beta(u(x_0))-\beta(v(x_0)) < 0$, which is a contradiction. So $x_0 \notin \partial\Omega$. Therefore $u(x_0) > 0$ and $v(x_0) > 0$. Since u and v are solutions of (1), we have in a neighbourhood $N(x_0)$ of x_0,

(2)
$$\begin{aligned}&-\Delta(u-v)\\ &\overset{a.e.}{=}\frac{1}{\varphi(x,u)\varphi(x,v)}[h(x)(\varphi(x,v)-\varphi(x,u))-P(u\varphi(x,v)-v\varphi(x,u))].\end{aligned}$$

Observe that $\int_{\partial N(x_0)} \frac{\partial(u-v)}{\partial n} \geq 0$. Thus $-\int_{N(x_0)} \Delta(u-v) \leq 0$. On the other hand, since φ is nondecreasing and concave down, we have $u(x_0)\varphi(x_0, v(x_0)) - v(x_0)\varphi(x_0, u(x_0)) < 0$. Then the integral of the right side of (2) over $N(x_0)$ is positive, which gives a contradiction. This shows the uniqueness of solution.

Next, to show the existence of a solution, we consider the Yosida approximation on β of equations (1):

$$\text{(3)}\qquad \begin{cases} -\varphi(x,u)\Delta u + Pu = h \\ u+\lambda\dfrac{\partial u}{\partial n} = (I+\lambda\beta)^{-1}u \qquad \text{on } \partial\Omega \end{cases}$$

where $\lambda > 0$. First we show *a priori* bound in a space $C^{1,\alpha}(\bar{\Omega})$ of every solution to (3), where $\alpha \in (0,1)$. We then look for a fixed point of the equation

$$\tag{4} \begin{cases} -\Delta u = \dfrac{h - Pv}{\varphi(x, v)} \\ u + \lambda \dfrac{\partial u}{\partial n} = (I + \lambda\beta)^{-1} v & \text{on } \partial\Omega. \end{cases}$$

Let u be a solution of equation (3) for $x \in \Omega$, i.e., u is a fixed point of equation (4). Let $G_{R,\lambda}$ be the Green's function such that

$$\begin{cases} -\Delta G = \delta \\ G + \lambda \dfrac{\partial G}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases}$$

Then we have

$$\tag{5} \begin{aligned} u(x) = &\int_\Omega G_{R,\lambda}(x, y)\left(\frac{h - Pu(y)}{\varphi(y, u(y))}\right) dy \\ &+ \int_{\partial\Omega} \frac{\partial}{\partial n}(G_{R,\lambda}(x, y))(I + \lambda\beta)^{-1} u(y) dy. \end{aligned}$$

Note that $\|Pu\|_\infty \leq \|h\|_\infty$ by a C^1-version general maximum principle. Now we estimate:

(6)

$$\begin{aligned} \int_\Omega \left| G_{R,\lambda} \frac{h - Pu}{\varphi} \right| &\leq \|G_{R,\lambda}\|_{L^{n/n-1}} \left\| \frac{h - Pu}{\varphi_0} \right\|_{L^n} \\ &< K_1 2 \|h\|_\infty \left\| \frac{1}{\varphi_0} \right\|_{L^n} = K_2 \|h\|_\infty. \end{aligned}$$

$$\tag{7} \begin{aligned} \int_\Omega \left| \frac{\partial G_{R,\lambda}}{\partial x_i} \frac{h - Pu}{\varphi} \right| &\leq N \int_\Omega \frac{1}{r^{n-1}} \left| \frac{h - Pu}{\varphi_0} \right| \leq N \left\| \frac{h - Pu}{\varphi_0} \right\|_{L^m} \int_\Omega \frac{1}{r^{\frac{(n-1)m}{m-1}}} \\ &< 2\|h\|_\infty \left\| \frac{1}{\varphi_0} \right\|_{L^m} \tilde{N} < K \|h\|_\infty \end{aligned}$$

since $\frac{1}{\varphi_0} \in L^m(\Omega)$ where $m > n$ (therefore $\frac{(n-1)m}{m-1} < n$), $\|Pu\|_\infty \leq \|h\|_\infty$, $\tilde{N} = \int \frac{1}{r^{\frac{(n-1)m}{m-1}}}$ and $r = \|x - y\|$. Here N is a constant independent of λ if λ is small by Lemma 1 (ii). Note that the second inequality follows from Hölder's inequality.

Next we notice that since $(I + \lambda\beta)^{-1}$ is increasing,

$$\begin{aligned} |(I+\lambda\beta)^{-1}u| &\leq \max\{|(I+\lambda\beta)^{-1}(-\|u\|_\infty)|, (I+\lambda\beta)^{-1}(\|u\|_\infty)\} \\ &= |(I+\lambda\beta)^{-1}(\|u\|_\infty)| \leq K_3\|u\|_\infty \leq K_4\|h\|_\infty \end{aligned} \tag{8}$$

by the assumption of monotonicity of the function $\beta \in C^2$ and $\beta(0) = 0$.

Since $G_{R,\lambda}(x, y)$ is a smooth function for $x \in \Omega$, $y \in \partial\Omega$, we have

$$\begin{aligned} \int_{\partial\Omega} \left| \frac{\partial(D_{x_i}G_{R,\lambda})}{\partial n}(I+\lambda\beta)^{-1}u(y) \right| dy &\leq C\|h\|_\infty \int_{\partial\Omega} \left(\left| \frac{\partial(D_{x_i}G_D)}{\partial n} \right| + 1 \right) \\ &\leq C_1\|h\|_\infty \end{aligned} \tag{9}$$

by Lemma 1 (ii) if λ is small. Thus (6), (7) and (9) imply that

$$\|D_{x_i}u\|_\infty \leq (K\|h\|_\infty + C_1\|h\|_\infty) = M\|h\|_\infty \tag{10}$$

where M is independent of λ since $\|G_{R,\lambda} - G_D\|_{C^{2,\alpha}} \to 0$ as $\lambda \to 0^+$, $0 \leq \lambda < 1$ by Lemma 1 (ii). Therefore, applying the elliptic regularity (see, for example, [1] and Theorem 13.1 (d) [7]) to (4) together with the fact that $\|\nabla u\|_\infty \leq M\|h\|_\infty$, we have $u \in W^{2,m}(\Omega)$ and

(11)

$$\|u\|_{W^{2,m}} \leq C\left(\left\| \frac{h - Pu}{\varphi_0} \right\|_{L^m} + \|(I+\lambda\beta)^{-1}u\|_{W^{1/2,m}(\partial\Omega)} + \|u\|_{W^{1,m}} \right).$$

Therefore by (8) and (10)

$$\|u\|_{W^{2,m}} \leq C_0\|h\|_\infty \tag{12}$$

for some generic constant C_0. Since $m > n$, the Sobolev imbedding theorem implies

$$\|u\|_{C^{1,\alpha}(\bar{\Omega})} \leq M_1 \|h\|_\infty \tag{13}$$

where some $\alpha \in (0,1)$. By the arguments used above, M_1 is a constant independent of P and λ.

Let $v \in C^{1,\alpha}(\bar{\Omega})$ and define $T : C^{1,\alpha}(\bar{\Omega}) \to C^{1,\alpha}(\bar{\Omega})$ as follows. Let $u = Tv$ be a solution of (4). We want to show that T has a fixed point for some $u \in C^{1,\alpha}(\bar{\Omega})$. To do this, we apply the following fixed-point theorem (see page 280 Theorem 11.3 in [3]): let T be a compact mapping of a Banach space E into itself, and suppose that there exists a constant K such that $\|u\|_E < K$ for all $u \in E$ and $\theta \in [0,1]$ satisfying $u = \theta T u$. Then T has a fixed point.

One can show that T is a compact operator by replacing u by v in estimates (6), (7) and (9). Consider equation $v = \theta T v$ for $\theta \in [0,1]$. Replacing T by θT to get an estimate corresponding to (13), it is not hard to show that $\|u\|_{C^{1,\alpha}(\bar{\Omega})} \leq \theta M_1 \|h\|_\infty \leq M_2 \|h\|_\infty$. So there exists a constant $K > 0$ such that $\|u\|_{C^{1,\alpha}(\bar{\Omega})} < K$. Thus by the above fixed-point theorem, $Tu = u$ for some $u \in C^{1,\alpha}(\bar{\Omega})$. We denote it by u_λ. Note that (3) is equivalent to

$$\begin{cases} -\varphi(x,u)\Delta u + Pu = h \\ -\dfrac{\partial u}{\partial n} = \beta(I + \lambda\beta)^{-1} u \qquad \text{on } \partial\Omega, \end{cases}$$

and u_λ is unique. (See the proof of uniqueness above.) By (12), $\|u_\lambda\|_{W^{2,m}(\Omega)}$ is uniformly bounded in λ where λ is small. Thus there exists a subsequence of $\{u_\lambda\}$, denoted by $\{u_\lambda\}$ again, such that $u_\lambda \overset{w}{\rightharpoonup} u$ in $W^{2,m}(\Omega)$. Also by (13), we have a subsequence of $\{u_\lambda\}$ such that $u_\lambda \to u$ in $C^{1,\tilde{\alpha}}(\bar{\Omega})$ for some $\tilde{\alpha} \in (0,\alpha)$. Thus we have, for $v \in$

$W^{-2,m'}(\Omega)$ where $m' = \frac{m}{m-1}$, $\langle -\Delta u_\lambda, v\rangle \to \langle -\Delta u, v\rangle$ and $\left\langle \frac{h-Pu_\lambda}{\varphi(x,u_\lambda)}, v\right\rangle$ $\to \left\langle \frac{h-Pu}{\varphi(x,u)}, v\right\rangle$ since $\varphi(x,u_\lambda) \to \varphi(x,u)$ in $C(\bar{\Omega})$. So $\langle \Delta u, v\rangle$ $= \left\langle \frac{h-Pu}{\varphi(x,u)}, v\right\rangle$. Therefore $-\Delta u = \frac{h-Pu}{\varphi(x,u)}$, and so u is a solution in $W^{2,m}(\Omega) \bigcap C^{1,\tilde{\alpha}}(\bar{\Omega})$ for some $\tilde{\alpha}$ such that $0 < \tilde{\alpha} < \alpha$.

The compactness of the solution operator S in $C(\bar{\Omega})$ such that $u = Sh$ follows from (12) and (13) for $u_\lambda \to u$ by the Ascoli-Arzéla theorem.

REMARK. One can say that $u \in W^{2,m}(\Omega)$ is a solution to (1) if

$$\int_\Omega \nabla u \cdot \nabla v \, dx = \int_\Omega \frac{Pu - h}{\varphi(x,u)} \cdot v \, dx$$

for all $v \in C_0^\infty(\Omega)$.

Let $F(x,\xi) \in C^1(\bar{\Omega} \times \mathbf{R})$ such that $F(x,0) = 0$ and $|F_\xi(x,\xi)| \leq M$ for $(x,\xi) \in \Omega \times \mathbf{R}$, where $M > 0$ is some constant. Consider the nonlinear elliptic problem

$$\begin{cases} -\varphi(x,u)\Delta u = F(x,u) \\ \dfrac{\partial u}{\partial n} + \beta(u) = 0 \qquad \text{on } \partial\Omega \end{cases} \tag{14}$$

where $\varphi \in G$ and β is strictly increasing, $\beta(0) = 0$.

We define upper and lower solutions for degenerate nonlinear elliptic equations.

Let $\varphi \geq \varphi_0 \geq 0$, $\frac{1}{\varphi_0} \in L^m(\Omega)$ where $m > n$. Let $u \in W^{2,m}(\Omega)$ $\bigcap C^1(\bar{\Omega})$.

(i) u is called an upper solution of (14) if u satisfies

$$\begin{cases} -\varphi(x,u)\Delta u \geq F(x,u) \qquad \text{a.e. in } \Omega \\ \dfrac{\partial u}{\partial n} + \beta(u) \geq 0 \qquad \text{on } \partial\Omega \end{cases}$$

(ii) u is a lower solution of (14) if u satisfies

$$\begin{cases} -\varphi(x,u)\Delta u \le F(x,u) & \text{a.e. in } \Omega \\ \dfrac{\partial u}{\partial n} + \beta(u) \le 0 & \text{on } \partial\Omega \end{cases}$$

Now we shall extend the results of the method of upper-lower solution to the case of a degenerate elliptic equation.

THEOREM. *Suppose u_0 and v_0 are upper and lower solutions, respectively, of (14) with $u_0 \ge v_0$ on $\bar{\Omega}$. Then there exists a maximal solution u of (14) such that $v_0 \le u \le u_0$.*

PROOF. We choose a constant P such that $P > \sup_{\bar{\Omega}\times[a,b]} |D_2 F(x,u)|$ where a and b are the minimum of v_0 and maximum of u_0, respectively, and D_2 denotes the derivative with respect to the second component. Define a mapping T by $u = Tv$, where u is the unique solution of

$$(15) \qquad \begin{cases} -\varphi(x,u)\Delta u + Pu = F(x,v) + Pv :\equiv h \\ \dfrac{\partial u}{\partial n} + \beta(u) = 0 \qquad \text{on } \partial\Omega, \end{cases}$$

where $v \in \mathbf{K} = C^+(\bar{\Omega})$. For the existence of a unique solution u, see Lemma 2. Note $h \ge 0$ by choice of P.

Next we claim that T is monotone. We need to show that if u_i, where $i = 1, 2$, are solutions to

$$(16) \qquad \begin{cases} -\varphi(x,u_i)\Delta u_i + Pu_i = F(x,v_i) + Pv_i :\equiv h_i \\ \dfrac{\partial u_i}{\partial n} + \beta(u_i) = 0 \qquad \text{on } \partial\Omega, \end{cases}$$

with $v_1 \ge v_2 \not\equiv v_1$, then $u_1 > u_2$. Notice that if $v_1 \ge v_2 \not\equiv v_1$, then $h_1 \ge h_2 \ge 0$ and $h_1 \not\equiv h_2$. Assume $u_1(x) \le u_2(x)$ for at least one x. Let $\min_{x\in\bar{\Omega}}(u_1(x)-u_2(x)) = u_1(x_0)-u_2(x_0) < 0$. Then as in the proof of uniqueness of the solution in Lemma 2, we can show that $x_0 \notin \partial\Omega$. Also we can use the same argument as in the proof of the uniqueness in Lemma 2. Take the integral $\int_{N(x_0)}$ on both sides of the equation $-\Delta(u_1-u_2) \overset{a.e.}{=} \frac{1}{\varphi(x,u_1)\varphi(x,u_2)}[h_1\varphi(x,u_2)-h_2\varphi(x,u_1)-P(u_1\varphi(x,u_2)-u_2\varphi(x,u_1))]$. Then the integral of the left side is nonpositive while the left side is positive. This contradiction shows $u_1 > u_2$.

Let u_0 be an upper solution of (14) and let $u = Tu_0$. Then we have

$$-\Delta(u-u_0) + \frac{P}{\varphi(x,u)}(u-u_0) \overset{a.e.}{=} \frac{F(x,u_0)}{\varphi(x,u)} + \Delta u_0 \leq 0. \tag{17}$$

Suppose $u(x) > u_0(x)$ for at least one $x \in \bar{\Omega}$.

Let $(u-u_0)(x_0) = \max_{x\in\bar{\Omega}}(u-u_0)(x) > 0$. Then $x_0 \notin \partial\Omega$ and since in the neighbourhood $N(x_0)$ of x_0, $\int_{\partial N(x_0)} \frac{\partial(u-u_0)}{\partial n} \leq 0$, we have $-\int_{(x_0)} \Delta(u-u_0) \geq 0$. So the integral of the left side over $N(x_0)$ in (17) is positive, while the left side is nonpositive, a contradiction. Thus $Tu_0 \leq u_0$.

Similarly, one can see that $Tv_0 \geq v_0$ if v_0 is a lower solution of (14). By the monotonicity of T 0and the inequalities $v_0 \leq Tv_0$ and $Tu_0 \leq u_0$, we have $Tv \in [[v_0, u_0]]$ for $v \in [[v_0, u_0]]$. Now recall that the nonlinear operator $(-\varphi(x,\cdot)\Delta + P)^{-1}$ under the nonlinear boundary condition is compact in $C^{1,\alpha}(\bar{\Omega})$ for some $\alpha \in (0,1)$. (See Lemma 2.) Also note that the order interval $[[v_0, u_0]]$ is a closed, bounded convex subset of $C(\bar{\Omega})$. Now apply the Schauder fixed-point theorem to get a fixed point $\tilde{u}$ of T. Therefore $\tilde{u}$ is a solution of (14).

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