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# A Geometric Characterization of Complete Continuity Property

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ABSTRACT. We introduce the notion of the mean Bocce dentable and provide the geometric characterization of the CCP. We show that X has the CCP if and only if every bounded subset of X is mean Bocce dentable.

## 1. Introduction and Preliminaries

If all bounded linear operators from $L_1$ into a Banach space $X$ are Dunford-Pettis (i.e.,weakly convergent sequences onto norm convergent sequences), then we say that $X$ has the complete continuity property (CCP). The CCP is a weakening of the Radon-Nikodym property(RNP). Basic results of Bourgain and Talagrand began to suggest the possibility that the CCP, like the RNP, can be realized as an internal geometric property of Banach spaces. The purpose of this paper is to provide such a realization. We introduce the notion of mean Bocce dentable and we show that $X$ has the CCP if and only if every bounded subset of $X$ is mean Bocce dentable. Throughout this paper, $X$ denotes an arbitrary Banach space, $X^*$ the dual space of $X$, $B(X)$ the closed unit ball of $X$, and $S(X)$ the unit sphere of $X$. The triple $(\Omega, \Sigma, \mu)$ refers to the Lebesgue measure space on $[0,1]$, $\Sigma^+$ to the sets in $\Sigma$ with positive measure, and $L_1$ to $L_1(\Omega, \Sigma, \mu)$. All notation and terminology, not otherwise explained, are as in [1].

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DEFINITION 1.1. For $f$ in $L_1$ and $A$ in $\Sigma$, the Bocce oscillation of $f$ on $A$ is given by

$$Bocce - oscf\mid_A \equiv \frac{\int_A \mid f - [\frac{\int_A f d\mu}{\mu(A)}] \mid d\mu}{\mu(A)}$$

Observing the convention that 0/0 is 0

DEFINITION 1.2. A subset $K$ of $L_1$ satisfies the Bocce criterion if for each $\varepsilon > 0$ and $B$ in $\Sigma^+$ there is a finite collection $F$ fo subsets of $B$ each with positive measure such that for each $f$ in $K$ there is an $A$ in $F$ satisfying

$$Bocce - oscf\mid_A < \varepsilon$$

The following theorem found in [2].

THEOREM 1.3. *A bounded linear operator $T$ from $L_1$ into $X$ is Dunford-Pettis if and only if the subset $T^*(B(X^*))$ of $L_1$ satisfies the Bocce criterion.*

Towards a martingale characterization of the CCP, fix an increasing sequence $\{\pi_n\}_{n\geq 0}$ of finite positive interval partitions of $\Omega$ such that $V\sigma(\pi_n) = \Sigma$ and $\pi_0 = \{\Omega\}$. Let $F_n$ denote the sub-$\sigma$-field $\sigma(\pi_n)$ of $\Sigma$ that is generated by $\pi_n$. For $f$ in $L_1(X)$, let $E_n(f)$ denote the conditional expectation of $f$ given $F_n$.

DEFINITION 1.4. A sequence $\{f_n\}_{n\geq 0}$ in $L_1(X)$ is an X-valued martingale with respect to $\{F_n\}$ if for each n we have that $f_n$ is $F_n$-measurable and $E_n(f_{n+1}) = f_n$ in $L_1$. The martingale $\{f_n\}$ is uniformly bounded provided that $sup_n \parallel f_n \parallel_{L_\infty}$ is finite.

There is a one-to-one correspondence between the bounded linear operators $T$ from $L_1$ into $X$ and the uniformly bounded X-valued martingale $\{f_n, F_n\}$.

THEOREM 1.5. *([3]) A bounded linear operator $T$ from $L_1$ into $X$ is Dunford Pettis if and only if the corresponding martingale is cauchy in the pettis norm.*

REMARK. A Banach space $X$ has CCP if and only if all uniformly bounded X-valued martingales are pettis-cauchy.

## 2. Main Results

Clearly, if X has the RNP. then $X$ has CCP. Both the Bourgain Rosenthal space [6] and the dual of the James tree space [7] have the CCP yet fail the RNP.

Dentability characterizations of the RNP are well-known ([1])

The following statements are equivalent

(1) X has the RNP

(2) Every bounded subset D of X is dentable

(3) Every bounded subset D of X is $\sigma$-dentable

DEFINITION 2.1. A subset D of X is weak-norm-one-dentable if for each $\varepsilon > 0$ there is a finite subset F of D such that for each $x^*$ in $S(X^*)$ there is x in F satisfying

$$x \notin \overline{co}\{z \in D : | x^*(z-x) | \geq \varepsilon\} \equiv \overline{co}(D - V_{\varepsilon,x^*}(x))$$

The following theorem is found in [2, 5], which is due to M. Girardi and M. Petrakis.

THEOREM 2.2. *X has the CCP if and only if every bounded subset of X is weak-norm-one dentable*

For characterization of the CCP, we introduce the following mean Bocce dentable

DEFINTION 2.3. A subset D of X is mean Bocce dentable if for each $\varepsilon > 0$ there is a finite subset F of D such that for each $x^*$ in

$S(X^*)$ there is x in F satisfying : if $x = \frac{1}{n}z_1 + \cdots + \frac{1}{n}z_n$ with $z_i \in D$, some $n$ then $| x^*(x - z_1) | \equiv | x^*(x - z_2) | \equiv \cdots \equiv | x^*(x - z_n) | < \varepsilon$.

We show that these dentability conditions provide an internal geometric characterization in the following theorems.

THEOREM 2.4. *If a bounded D of X is not mean Bocce dentable, then there is an increasing sequence* $\{\pi_n\}$ *of partitions of* $[0, 1)$ *and a D-valued nartingale* $\{f_n, \sigma(\pi_n)\}$ *that is not cauchy in the pettis norm. Moreover,* $\{\pi_n\}$ *can be chosen so that* $V\sigma(\pi_n) = \Sigma, \pi_0 = \{\Omega\}$, *and each* $\pi_n$ *partition* $[0, 1)$ *into a finite number of half-open intervals.*

PROOF. Let $D$ be a subset of $X$ that is not mean Bocce dentable. According there is an $\varepsilon > 0$ satisfying :

For each finite subset $F$ of $D$ there is $x_F^*$ in $S(X^*)$ such that each $x$ in $F$ has the $x = \sum_{i=1}^{m} \frac{1}{m} z_i$ with $| x^*(x - z_1) | \equiv \cdots \equiv | x^*(x - z_m) | \geq \varepsilon$ for a suitable choice of $z_i \in D$, some $m$.

We shall use property $(*)$ to construct an increasing sequence $\{x_n^*\}_{n \geq 1}$ in $S(X^*)$ such that for each nonnegative n :

(1) $f_n$ has the form $f_n = \sum_{E \in \pi_n} x_E \, \chi_E$ where $x_E$ is in $D$

(2) $\int_\Omega | x_{n+1}^*(f_{n+1} - f_n) | \, d\mu \geq \varepsilon$

(3) if $E$ is in $\pi_n$, then $E$ has the form $[a, b)$ and $\mu(E) < \frac{1}{2^n}$

(4) $\pi_0 = \{\Omega\}$.

Condition (3) guarantee that $V\sigma(\pi_n) = \Sigma$ while condition (2) guarantee that $\{f_n\}$ is not cauchy in the pettis norm. Towards construction, pick an arbitrary $x$ in $D$. Set $\pi_0 = \{\Omega\}$ and $f_0 = x\chi_\Omega$. Fix $n \geq 0$. Suppose that a partition $\pi_n$ of $\Omega$ consisting of intervals of length at most $\frac{1}{2^n}$ and a function $f_n = \sum_{E \in \pi_n} x_E \chi_E$ with $x_E \in D$ have been constructed. We now construct $f_{n+1}, \pi_{n+1}$ and $x_{n+1}^*$ satisfying (1), (2) and (3). Apply (*) to $F = \{x_E : E \in \pi_n\}$ and find the associated $x_F^* = x_{n+1}^*$ in $S(X^*)$. Fix an element $E = [a, b)$ of $\pi_n$ we first define $f_{n+1}\chi_E$. Property (*) gives that $x_E$ has the form

$$x_E = \frac{1}{m}\sum_{i=1}^{m} x_i \text{ with } \mid x_{n+1}^*(x_E - x_1) \mid \equiv \cdots \equiv \mid x_{n+1}^*(x_E - x_m) \mid \geq \varepsilon$$

for suitable choice of $x_i \in D$. Using repetition we arrange to have $\frac{1}{m} < \frac{1}{2^{n+1}}$. It follows that there are real numbers $d_0, d_1, \cdots, d_m$ such that

$$a = d_0 < d_1 < \cdots < d_{m-1} < d_m = b$$

and

$$d_i - d_{i-1} = \frac{1}{m}(b-a) \text{ for } i = 1, 2, \cdots, m$$

Set

$$f_{n+1}\chi_E = \sum_{x=1}^{m} x_i \chi_{[d_{i-1}, d_i)}$$

Define $f_{n+1}$ on all of $\Omega$ similarly. Let $\pi_{n+1}$ be the partition consisting of all the intervals $[d_{i-1}, d_i)$ obtained from letting $E$ range over $\pi_n$. Clearly, $f_{n+1}$ and $\pi_{n+1}$ satisty conditions (1) and (2). We have

$$\begin{aligned}\int_E \mid x_{n+1}^*(f_{n+1} - f_n) \mid d\mu &= \sum_{i=1}^{m} \int_{d_{i-1}}^{d_i} \mid x_{n+1}^*(x_i - x_E) \mid d\mu \\ &= \frac{b-a}{m} \sum_{i=1}^{m} \mid x_{n+1}^*(x_i - x_E) \mid \\ &\geq \mu(E) \cdot \varepsilon\end{aligned}$$

Hence, $\int_\Omega \mid x_{n+1}^*(f_{n+1} - f_n) \mid d\mu \geq \varepsilon \Sigma \mu(E) = \varepsilon$. To insure that $\{f_n\}$ is indeed a martingale. Fix $E = [a, b)$ in $\pi_n$. Using the above notation, we have for at most all $t$ in $E$,

$$\begin{aligned} E_n(f_{n+1})(t) &= \frac{1}{b-a}\int_a^b f_{n+1}d\mu \\ &= \frac{1}{b-a}\sum_{i=1}^{m}\int_{d_{i-1}}^{d_i} f_{n+1}d\mu \\ &= \sum_{i=1}^{m}(\frac{d_i - d_{i-1}}{b-a})x_i \\ &= \sum_{i=1}^{m}(\frac{1}{m}x_i) = x_E = f_n(t) \end{aligned}$$

Thus $E_n(f_n) = f_n$ a.e, this completes the necessary constructions.

THEOREM 2.5. *([3]) If $A$ is in $\Sigma^+$ and $f$ in $L_\infty(\mu)$ is not contant a.e. on $A$, then there is an increasing sequence $\{\pi_n\}$ of positive finite measurable partitions of $A$ such that $V\sigma(\pi_n) = \Sigma \cap A$ and for each n*

$$\mu(\cup\{E : E \in \pi_n, \frac{\int_E f d\mu}{\mu(E)} \geq \frac{\int_A f d\mu}{\mu(A)}\}) = \frac{\mu(A)}{2}$$

*and so*

$$\mu(\cup\{E : E \in \pi_n, \frac{\int_E f d\mu}{\mu(E)} < \frac{\int_A f d\mu}{\mu(A)}\}) = \frac{\mu(A)}{2}$$

THEOREM 2.6. *If all bounded subsets of X are mean Bocce dentable, then X has the complete continuity poperty.*

PROOF. Let all bounded subsets of X be mean Bocce dentable and let $F : L_1 \to X$ be a bounded linear operator. We shall show that the subset $T^*(B(X^*))$ of $L_1$ satisfies the Bocce criterion. Then By Theorem 1.3,. X has the complete comtinuity property. To this end, fix $\varepsilon > 0$ and B in $\Sigma^+$. Let F denote the vector measure from $\Sigma$ into X given by $F(E) = T(\chi_E)$. Since the subset $\{\frac{F(E)}{\mu(E)} : E \subset B, E \in \Sigma^+\}$

of $\chi$ is bounded, it is mean Bocce dentable. Accordingly, there is a finite collection F of subsets of B each in $\Sigma^+$ such that for each $x^*$ in the unit ball of $X^*$ there is a set A in F such that if

$$\frac{F(A)}{\mu(A)} = \frac{1}{m}\sum_{i=1}^{m}\frac{F(E_i)}{\mu(E_i)}$$

for some subsets $E_i$ of B with $E_i \in \Sigma^+$, then

$$\frac{1}{m}\sum_{i=1}^{m} \mid \frac{x^*F(E_i)}{\mu(E_i)} \mid < \varepsilon$$

Fix $x^* \in B(X^*)$ and find the associated A in F. By definition, the set $T^*(B(X^*))$ will safisfy the Bocce criterion provided that $Bocce-osc(T^*x^*) \mid_A \leq \varepsilon$. If $T^*x^* \in L_1$ is constant a.e on A then the $Bocce-osc(T^*x^*) \mid_A = 0$, we are finished. So assume $T^*x^*$ is not constant a.e on A.

For a finite positive measurable partition $\pi$ of $A$, denote

$$f_\pi = \sum_{E\in\pi}\frac{F(E)}{\mu(E)}\chi_E$$

and

$$E_\pi^+ = \cup\{E \in \pi : \frac{x^*F(E)}{\mu(E)} \geq \frac{x^*F(A)}{\mu(A)}\}$$

and

$$E_\pi^- = \cup\{E \in \pi : \frac{x^*F(E)}{\mu(E)} < \frac{x^*F(A)}{\mu(A)}\}$$

Note that for E in $\Sigma$

$$x^*F(E) = \int_E x^*T^*d\mu$$

Compute

$$\int_A | x^* f_\pi - \frac{x^*F(A)}{\mu(A)} | \, d\mu = \mu(A)[\frac{\mu(E_\pi^+)}{\mu(A)} | \frac{x^*F(E_\pi^+)}{\mu(E_\pi^+)} - \frac{x^*F(A)}{\mu(A)} | + \frac{\mu(E_\pi^-)}{\mu(A)} | \frac{x^*F(E_\pi^-)}{\mu(E_\pi^-)} - \frac{x^*F(A)}{\mu(A)} |]$$

Since the $L_1$-function $T^*x^*$ is bounded, for now we may view $T^*x^*$ as an element in $L_\infty$. For applying Theorem 2.5 to A with $f \equiv T^*x^*$ produces an increasing sequence $\{\pi_n\}$ of positive measurable partition of A satisfying

$$V\sigma(\pi_n) = \Sigma \cap A, \mu(E_{\pi_n}^+) = \frac{\mu(A)}{2} = \mu(E_\pi^-)$$

Since

$$\frac{\mu(A)}{2} = \frac{\mu(A)}{2m} + \cdots + \frac{\mu(A)}{2m}$$

For $\pi = \pi_n$, condition (2) becomes

$$\begin{aligned} &\int_A | x^* f_\pi - \frac{x^*F(A)}{\mu(A)} | \, d\mu \\ &= \mu(A)[\frac{1}{2m} | \frac{x^*F(E_\pi^+)}{\mu(E_\pi^+)} - \frac{x^*F(A)}{\mu(A)} | + \cdots + \frac{1}{2m} | \frac{x^*F(E_\pi^+)}{\mu(E_\pi^+)} - \frac{x^*F(A)}{\mu(A)} | \\ &+ \frac{1}{2m} | \frac{x^*F(E_\pi^-)}{\mu(E_\pi^-)} - \frac{x^*F(A)}{\mu(A)} | + \cdots + \frac{1}{2m} | \frac{x^*F(E_\pi^-)}{\mu(E_\pi^-)} - \frac{x^*F(A)}{\mu(A)} |] < \varepsilon \end{aligned}$$

Since $\frac{F(A)}{\mu(A)}$ has the form

$$\frac{F(A)}{\mu(A)} = \frac{1}{2}\frac{F(E_{\pi_n}^+)}{\mu(E_{\pi_n}^+)} + \frac{1}{2}\frac{F(E_{\pi_n}^-)}{\mu(E_{\pi_n}^-)}$$

Applying property (1) to equation (3) yields that for each $\pi_n$

$$\int_A | x^* f_{\pi_n} - \frac{x^*F(A)}{\mu(A)} | \, d\mu < \mu(A) \, \varepsilon$$

Since $V\sigma(\pi_n) = \Sigma \cap A$ and

$$\begin{aligned}(x^* f_{\pi_n}) \mid_A &= \sum_{E\in\pi_n} \frac{x^*F(E)}{\mu(E)}\chi_E \\ &= \sum_{E\in\pi_n} \frac{\int_E (T^*x^*)d\mu}{\mu(E)} \\ &= E_{\pi_n}(T^*x^*) \mid_A .\end{aligned}$$

We have that $(x^* f_{\pi_n}) \mid_A$ converges to $(T^*x^*) \mid_A$ in $L_1$-norm. Hence

$$Bocce - osc(T^*x^*) \mid_A \equiv \frac{\int_A \mid (T^*x^*) - [\frac{\int_A (T^*x^*)d\mu}{\mu(A)}]d\mu}{\mu(A)} \leq \varepsilon$$

Thus $T^*(B(x^*))$ satisfies the Bocce crterion, and X has the complete continuity property.

THEOREM 2.7. *The following statements are equivalent*
*(1) X has CCP*
*(2) Every bounded subset of X is weak-norm-one-dentable*
*(3) Every bounded subset of x is mean Bocce dentable*

PROOF. From Theorem 2.2, we show that (1), (2) are equivalent and it is clear from the definition that (2) implies (3). Also, (1) implies (3) follows from Theorem 2.4., Theorem 1.5., Theorem 2.6 show that (3) implies (1).

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