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On the Stability Theory in Dynamical Systems

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ABSTRACT. In this paper, we show that a continuous function $f : X \to X$ has regular coordinate then (X, f) has some properties which are similar to results following from a hypothesis of hyperbolicity.

1. Introduction

Let (X, d) be a compact metric space and $f : X \to X$ be a continuous surjective mapping. Denote by $S_f(X), S_{\bar{f}}(X)$ the set of all f-orbits of X and the set of all backward f-orbits of X,respectively. Let $\varepsilon > 0$. We define the local stable set $W^s_\varepsilon(x)$ for $x \in X$ by

$$W^s_\varepsilon(x) = \{z \in X : d(f^n(x), f^n(z)) \leq \varepsilon \quad \text{for} \quad n \geq 0\}$$

and the local unstable set $W^u_\varepsilon(\{y_{-i}\})$ for $\{y_{-i}\} \in S_{\bar{f}}(X)$ by

$$W^u_\varepsilon(\{y_{-i}\}) = \{z \in X : \quad \text{there is} \quad \{z_{-i}\} \in S_{\bar{f}}(X) \quad \text{with} \quad z_0 = z, \\ d(y_{-i}, z_{-i}) \leq \varepsilon \text{ for } i \geq 0\}$$

We say that f has regular coordinate if there exists $\alpha > 0$ with the property that for every $0 < \varepsilon < \alpha$ there is a $\delta = \delta(\varepsilon) > 0$ such that for every $x \in X$ and every $\{y_{-i}\} \in S_{\bar{f}}(X)$, $d(x, y_0) \leq \delta$ implies that

$$W^s_\varepsilon(x) \cap W^u_\varepsilon(\{y_{-i}\}) = \quad \text{singleton}$$

In this paper, assume that f has regular coordinate and we will show that (X, f) has some properties which are similar to results following from a hypothesis of hyperbolicity.

Basic terminologies are followed from [1] and [4].

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2. Basic results

LEMMA 2.1. *Let $f : X \to X$ have regular coordinate and $\{x_i\}, \{y_i\} \in S_f(X)$. Then there exists $c > 0$ such that if $d(x_i, y_i) \leq c$ for every $i \in Z$ then $x_0 = y_0$.*

PROOF. Let $c < \min\{\alpha, \delta(\alpha)\}$ and $d(x_i, y_i) \leq c$ for all $i \in Z$. Then $d(x_0, y_0) < c \leq \delta$ and $x_0, y_0 \in W^s_\alpha(x) \cap W^u_\alpha(\{y_{-i}\})$. So we have $x_0 = y_0$.

So we call f has *regular coordinate with expansive constant* c. Hereafter, assume f has regular coordinate with expansive constant c.

LEMMA 2.2 [1]. *For each $r \geq 0$, there exists an integer $N_r > 0$ such that*

(i) *$f^n(W^s_c(x)) \subset W^s_r(f^n(x))$ for all $x \in X$ and $n \geq N_r$*
(ii) *If $d(x_{-i}, y_{-i}) \leq c$ ($\{x_{-i}\}, \{y_{-i}\} \in S_{\overline{f}}(X)$), for every $i \geq 0$, then $d(x_{-n}, y_{-n}) \leq r$ for all $n \geq N_r$.*

For $x \in X$ and $\{x_{-i}\} \in S_{\overline{f}}(X)$ the *stable* and *unstable* sets are defined by

$$W^s(X,d) = \{y \in X : \lim_{n\to\infty}(d(f^n(x), f^n(y)) = 0\}$$

$$W^u(\{x_{-i}\}) = \{y : \text{there is } \{y_{-i}\} \in S_{\overline{f}}(X) \text{ with } y_0 = y \text{ such that } \lim_{n\to\infty}(x_{-n}, y_{-n}) = 0\}.$$

PROPOSITION 2.3. *Let $\varepsilon > 0$ be the number less than c, then*

(i) $W^s(x) = \bigcup_{n\geq 0}\{y_{-n} : y_0 \in W^s_\varepsilon(f^n(x))$ *and* $y_{-n} \in \{y_{-i}\} \in S_{\overline{f}}(y_0)\}$
(ii) $W^u(\{x_{-i}\}) = \bigcup_{n\geq 0}\{y_n : y_n = f^n(y_0), \quad y_0 \in W^u_\varepsilon(\{x_{-n-i}\})$.

PROOF. (i) Let $y = y_{-k}$ for some $\{y_{-i}\} \in S_{\overline{f}}(y_0)$ with $f^k(y_{-k}) = y_0 \in W^s_\varepsilon(f^k(x))$.

For $r > 0$, there exists $N = N_r$ such that for all $n \geq N$

$$d(f^{k+n}(x), f^k(y_0)) \leq r.$$

Hence we have $d(f^n(x), f^n(y_{-k})) \leq r$ for all $n \geq N+k$. Thus $d(f^n(x), f^n(y_{-k})) \to 0$ as $n \to \infty$ and so $y = y_{-k} \in W^s(x)$.

Conversely, let $z \in W^s(x)$. For $\varepsilon > 0$, there exists an integer N such that for all $n \geq N, d(f^n(x), f^n(z)) \leq \varepsilon$ and so $f^N(z) \in W^s_\varepsilon(f^N(x))$. Let $f^N(z) = y_0$. We can choose $\{y_{-i}\} \in S_{\overline{f}}(y_0)$ with $y_{-N} = z$. So $f^N(z) = y_0 \in W^s_\varepsilon(f^N(x))$ and therefore

$$z = y_{-N} \in \bigcup_{n \geq 0} \{y_{-n} : \{y_{-i}\} \in S_{\overline{f}}(X) \quad \text{and} \quad y_0 \in W^s_\varepsilon(f^n(x))\}.$$

(ii) Let $z \in \bigcup_{n \geq 0} \{y_n : y_0 \in W^u_\varepsilon(\{x_{-n-i}\})$. Then $z = y_k$, and $\{y_{-i}\} \in S_{\overline{f}}(X)$, $y_0 \in W^u_\varepsilon(\{x_{-k-i}\})$ for some $k \geq 0$. Since for every $i \geq 0, d(y_{-i}, x_{-k-i}) \leq \varepsilon < c$, there exists an integer $N > 0$ such that for every $n \geq N$, $d(y_{-n}, x_{-n-k}) \leq r$. Hence, for every $i \geq N + k$, we have $d(y_{k-i}, x_{-i}) \leq r$ and this means $z = y_k = f^k(y_0) \in W^s(x)$.

Conversely, Let $z \in W^u(\{x_{-i}\})$. Then there exists $\{z_{-i}\} \in S_{\overline{f}}(z)$ such that $d(x_{-i}, z_{-i}) \to 0$ as $i \to \infty$. So, for $\varepsilon > 0$, there exists N such that for every $n \geq N, d(x_{-n}, z_{-n}) \leq \varepsilon$. Let $z_{-N} = y_0$. Then $z_{-N-i} = y_{-i}$ and $z = y_n$. So we have $y_0 \in W^u_\varepsilon(\{x_{-n}\})$.

LEMMA 2.4. *If given $\varepsilon > 0$, there exists an integer $N \geq 1$ such that*

$$d(f^n(x), f^n(y)) \leq c, \quad 0 \leq n \leq N \qquad \text{and}$$

$$d(x_{-i}, y_{-i}) \leq c, \quad -N \leq n \leq 0 \qquad \textit{for some } \{x_{-i}\}, \{y_{-i}\} \textit{ with}$$

$$x_0 = x, y_0 = y,$$

then $d(x,y) < \varepsilon$.

PROOF. Assume the contrary. Let there exists an $\varepsilon_0 > 0$, such that for all $k \geq 1$, there exist $x^k, y^k \in X$ such that

$$d(f^n(x^k), f^n(y^k)) \leq c,\ 0 \leq n \leq k$$
$$d(x^k_{-n}, y^k_{-n}) \leq c,\ -k \leq n \leq 0 \quad \text{for some} \quad \{x^k_{-n}\}, \{y^k_{-n}\} \in S_{\overline{f}}(X)$$
$$d(x^k, y^k) \geq \varepsilon_0.$$

Asuume that $x^k \to x$, $y^k \to y, x^k_{-i} \to x_{-i}$, and $y^k_{-i} \to y_{-i}$ without loss of generality. Then $f(x^k_{-n-1}) = x^k_{-n} \to f(x_{-n-1}) = x_{-n}$ and $f(y^k_{-n-1}) = y^k_{-n} \to f(y_{-n-1}) = y_{-n}$. Thus we have $d(f^n(x), f^n(y)) \leq c$ and $d(x_{-n}, y_{-n}) \leq c$ for every $n \geq 0$. By Lemma 1, we have $x_0 = y_0$, but this is a contradiction.

Here, we give a topology on $S_{\overline{f}}(X)$ induced by the metric ρ on $S_{\overline{f}}(X)$ by

$$\rho(\{x_{-i}\}, \{y_{-i}\}) = \sup_{i \geq 0}\{d(x_{-i}, y_{-i})\}$$

Then clearly, $\rho(\{x_{-i}\}, \{y_{-i}\}) \geq d(x_{-n}, y_{-n})$ for every $n \geq 0$.

Let $\varepsilon > 0$ be given. Then, for some $\delta > 0$ we can define a function

$$a : K_\delta(X \times S_{\overline{f}}(X)) = \{(x, \{y_{-i}\}) \in X \times S_{\overline{f}}(X) : d(x, y_0) \leq \delta\} \to X$$

given by

$$a(x, \{y_{-i}\}) = W^s_\varepsilon(x) \cap W^u_\varepsilon(\{y_{-i}\})$$

THEOREM 2.5. *a is continuous.*

PROOF. Let $0 < \varepsilon < \frac{1}{3}c$, $(x, \{y_{-i}\}) \in K_\delta(X \times S_{\overline{f}}(X))$ and δ be as above. Let $r > 0$ be given and consider a neighborhood $B_r(a(x, \{y_{-i}\}),$

$r > 0$ of $a(x, \{y_{-i}\})$. Let $(u, \{v_{-i}\}) \in B_\delta(x, \{y_{-i}\}) = \{(z, \{w_{-i}\}) : d(x, z) < \delta$ and $\rho(\{y_{-i}\}, \{w_{-i}\}) < \delta\} \cap K_\delta$. For given $\epsilon > 0$ there exists $N = N(\epsilon)$ as in Lemma 2.4 and $l = l(N), (0 < l < \delta)$ such that

$$(1) \quad d(x_1, x_2) < l \ \text{ implies } \ d(f^n(x_1), f^n(x_2) < \epsilon \ \text{ for } \ 0 \le n \le N$$

We have for $(u, \{v_{-i}\}) \in B_l(x, \{y_{-i}\})$,

$$\begin{aligned}
&d(f^n(a(x, \{y_{-i}\}), f^n(a(u, \{v_{-i}\}) \\
&\le d(f^n(a(x, \{y_{-i}\}), f^n(x)) + d(f^n(x), f^n(u)) + d(f^n(u), f^n a(u, \{v_{-i}\})) \\
&\le 2\varepsilon + d(f^n(x), f^n(u)), \\
&d(a(x, y_{-i})_{-n}, a(u, \{v_{-i}\})_{-n}) \\
&\le d(a(x, \{y_{-i}\})_{-n}, y_{-n}) + d(y_{-n}, v_{-n}) + d(v_{-n}, a(u, \{v_{-n}\})_{-n}) \\
&\le 2\epsilon + \rho(\{y_{-i}\}, \{v_{-i}\}) \\
&\le 2\epsilon + \delta < 3\epsilon
\end{aligned}$$

for every $n \ge 0$. In particular, by (1), we get

$$d(f^n(a(x, \{y_{-i}\}), f^n(a(u, \{v_{-i}\})) \le 3\epsilon$$
$$d(a(x, \{y_{-i}\})_{-n}, a(u, \{v_{-i}\})_{-n}) \le 3\epsilon$$

for $0 \le n \le N$, which means $a(u, \{v_{-i}\}) \in B_r(a(x, \{y_{-i}\}))$. So the mapping a is continuous.

PROPOSITION 2.6.

(i) $W^s_\varepsilon(x) \cap B(x, \delta) = \{y : y = a(x, \{y_{-i}\})\}$, *for some* $\{y_{-i}\} \in \{S_{\overline{f}}(X),\ d(x, y) < \delta\}$

(ii) $W^u_\varepsilon(\{x_{-i}\}) \cap B(x, \delta) = \{y : y = a(y, \{x_{-i}\}),\ d(x, y) < \delta\}$

PROOF. The proof is straightforward and thus omitted.

PROPOSITION 2.7. *For the function a the followings are valid.*

$$a(x,\{x_{-i}\}) = x, \qquad a(a(x,\{y_{-i}\}),\{z_{-i}\}) = a(x,\{z_{-i}\})$$
$$a(x,a(y,\{z_{-i}\})) = a(x,\{z_{-i}\}), \quad f(a(x,\{y_{-i}\}) = a(f(x),f(\{y_{-i}\})$$

when the two sides of these relations are defined.

PROOF. It is clear that $a(x,\{x_{-i}\}) = x$. Let $3\epsilon < c$ and $v = a(a(x,\{y_{-i}\}),\{z_{-i}\})$ and $a(x,\{y_{-i}\}) = w$. Since $w \in W^s_\epsilon(x) \cap W^u_\epsilon(\{y_{-i}\})$ we have $v \in W^s_{2\epsilon}(x) \cap W^u_\epsilon(\{z_{-i}\})$ and by Lemma 2.1 we get $v = a(x,\{z_{-i}\})$. It is easily checked that $f(a(x,\{y_{-i}\}) = a(f(x),f(\{y_{-i}\})$ using the uniform continuity of f.

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