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On the Topological Stability in Dynamical Systems

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ABSTRACT. In this paper, we show that a persistent dynamical system is structurally stable with respect to $E_\alpha(X)$ for every $\alpha > 0$ if it is expansive. Also, we prove that a homeomorphism $f : \Omega(f) \to \Omega(f)$ has the semi-shadowing property then so does $f : \overline{C(f)} \to \overline{C(f)}$.

1. Introduction

P. Walters [5] studied the relationship between the pseudo-orbit tracing property and the topological stability of expansive homeomorphisms. Also, Lewowicz [2] defined the persistence, an weak form of topological stability of homeomorphisms on compact Riemannian manifolds.

In this paper, we gives a necessary condition for a persistent dynamical system to be structurally stable Also, we show that a homeomorphism $f : \Omega(f) \to \Omega(f)$ has the semi-shadowing property, then so does $f : \overline{C(f)} \to \overline{C(f)}$.

We consider homeomorphisms acting on a compact metric space. We let X denote a compact Riemannian manifold with a metric d and $\dim X \leq 2$. Let $H(X)$ denote the collection of all homeomorphisms of X to itself topologised by the C^0-metric

$$d_0(f, g) = \sup\{d(f(x), g(x)) | x \in X\}.$$

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$f \in H(X)$ is said to be *structurally stable* (with respect to $F \subset H(X)$) if for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that if $d_0(f,g) < \delta$ ($g \in F$), then there exists a $h \in H(X)$ satisfying $hg = fh$. $f \in H(X)$ is called *persistent* if for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that if $d_0(f,g) < \delta$ and $x \in X$, then there is $y \in X$ satisfying $d(f^n(x), g^n(y)) < \varepsilon$ for all $n \in \mathbb{Z}$. Let $Y \subset X$. $f \in H(Y)$ has the *semi-shadowing property* if for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that for each $x \in Y$ and $g \in H(Y)$ with $d_0(f,g) < \delta$, there exists $y \in Y$ such that $d(f^n(y), g^n(x)) < \varepsilon$ for every $n \in \mathbb{Z}$.

We say that $f \in H(X)$ is *expansive* if there exists $e(f) > 0$ such that if $d(f^n(x), g^n(y)) \leq e(f)$ for every $n \in \mathbb{Z}$, then $x = y$. Such numbers $e(f)$ are called *expansive constant* for f. Let $E_\alpha(X)$ denote the set of expansive homeomorphisms of X to itself with expansive constant α.

For $f \in H(X)$, define the recurrent set and the nonwandering set of f by

$$C(f) = \{x \in X : x \in \omega_f(x) \cap \alpha_f(x), \text{ where } \omega_f(x) \text{ and } \alpha_f(x) \text{ denote the positive and negative limit set of } x \text{ for } f, \text{ respectively}\},$$

$$\Omega(f) = \{x \in X : \text{for every neighborhood } U \text{ of } x \text{ and integer } n_0 > 0 \text{ there is } n \geq n_0 \text{ such that } f^n(U) \cap U \neq \emptyset\}.$$

A sequence of points $\{x_i\}_{i=a}^{b}, (-\infty \leq a < b \leq \infty)$ in X is called a *δ-pseudo-orbit* of f if $d(f(x_i), x_{i+1}) < \delta$ for $a \leq i \leq b$. A finite δ-pseudo-orbit $\{x_0, x_1, \cdots, x_n\}$ is called a δ-pseudo-orbit from x_0 to x_n. Given $x, y \in X, x$ is α-related to y in $A \subset X$ means there are α-pseudo-orbits from x to y and y to x in A. $O_f(x)$ denote the orbit of x for f and $B(x,\varepsilon)$ denote $\{y \in X : d(x,y) < \varepsilon\}$. For closed subsets A, B of $X, d(A,B) = \inf\{d(a,b) : a \in A, b \in B\}$.

Basic terminologies are followed from [1].

2. Basic results.

LEMMA 2.1 ([4]). *Let X be a compact topological manifold of dim ≥ 2 with metric d, and let $\varepsilon > 0$ be arbitrary. Then there exist $\delta = \delta(\varepsilon) > 0$ such that if $\{(x_1, y_1), \cdots, (x_n, y_n)\}$ is a finite set of points in $X \times X$ satisfying :*

(i) *for each $i = 1, \cdots, n$, $d(x_i, y_i) < \delta$; and*

(ii) *if $i \neq j$, then $x_i \neq x_j$, and $y_i \neq y_j$;*

then there is $h \in H(X)$ with $d_0(h, 1_X) < \varepsilon$ and $h(x_i) = y_i$ for $i = 1, \cdots, n$.

PROPOSITION 2.2. *If $f \in H(X)$ has semi-shadowing property, then so does f^k for every $k > 0$*

PROOF. Let $\varepsilon > 0$ be given. Let $\delta_1 = \delta_1(\varepsilon)$ be a number with the property of the semi-shadowing property and $\delta = \delta(\delta_1)$ as in Lemma 2.1. Let $d_0(f^k, g) < \delta$ and $x \in X$. For each positve integer l, consider $\{g^{-l}(x), g^{-l+1}(x), \cdots, g^l(x)\}$. Let define a set $\{(w_i, v_i)\}$ in $X \times X$ as follows.

$$\{(g^{-l}(x), g^{-l}(x)), (f(g^{-l}(x)), f(g^{-l}(x))), \cdots, (f^{k-1}(g^{-l}(x)), f^{k-1}(g^{-1}(x))), (f^k(g^{-l}(x)), g^{-l+1}(x)), (g^{-l+1}(x), g^{-l+1}(x)), \cdots, (x, x), \cdots, (f^{k-1}(x), f^{k-1}(x)), (f^k(x), g(x)), \cdots, (f^k(g^{l-1}(x)), g^l(x))\}.$$

Then the set $\{(w_i, v_i)\}$ satisfy the hypothesis of Lemma 2.1. So there exists $h \in H(X)$ with

$$d_0(h, 1_X) < \delta, \quad \text{and} \quad h(w_i) = v_i$$

Put $a = h \circ f$. Then we have

$$d(a(x), f(x)) = d(h \circ f(x), f(x)) < \delta_1$$

Hence, there exists y_l in X such that $d(f^n(y_l), a^n(x)) < \varepsilon$ for every integer n. Let $z_l = f^{kl}(y_l)$. Then $d(f^{nk}(z_0), g^n(x)) < \varepsilon$ for every $n, (-1 \leq n \leq 1)$. Let $z_0 \to z$ as $l \to \infty$. Then $d(f^{nk}(z), g^n(x)) \leq \varepsilon$ for every $n \in Z$. So f^k has the semi-shadowing property and the proof is complete.

THEOREM 2.2. *If $f \in H(X)$ has the semi-shadowing property, then f has the shadowing property.*

PROOF. Let $\varepsilon > 0$ be arbitrary. select numbers $\delta = \delta_1(\varepsilon)$ satisfying the semi-shadowing property and $\delta = \delta(\delta_1)$ as in Lemma 2.1 Let x_i , $i \in Z$ be a δ - pseudo-orbit with $x_i \neq x_j, i \neq j$. consider the finite pseudo-orbit $A_k = \{x_{-k}, \cdots, x, \cdots, x_k\}$.

Then the set

$$\{(f(x_{-k}), x_{-k+1}), (f(x_{-k+1}), x_{-k+2}), \cdots, (f(x_{k-1}), x_k)\}$$

satisfies the hypothesis of Lemma 2.1. Then there exists a $h \in H(X)$ such that $d_0(h, 1_X) < \delta$, and $h(f(x_i)) = x_{i+1}$, $i = -k, dots, k-1$. Let $g(x) = h \circ f(x)$, Then we have $d_0(f, g) < \delta_1$. So for the point x_{-k}, there is a y_k, such that $d(f^n(y_k), x_{-k+n}) < \varepsilon$ for $0 \leq n \leq k$. Let $f^k(y_k) = z_k$. Then we have $d(f^i(z_k), x_i) < \varepsilon$ for $-k \leq i \leq k$. Suppose $z_k \to z$ as $k \to \infty$. Then it is easy to show that $d(f^i(z), x_i) \leq \varepsilon$ for all $i \in Z$. so f has the shadowing property

THEOREM 2.3. *A persistent dynamical system is structurally stable with respect to $E_\alpha(x)$ for every $\alpha > 0$ if it is expansive.*

PROOF. Let $f \in H(X)$ be persistent and let $e(f)$ be an expansive constant for f. Choose sufficiently small $\varepsilon > 0$ satisfying $\varepsilon < \min\{\frac{1}{4}e(f), \frac{1}{4}\alpha\}$. Let $\delta = \delta(\frac{1}{2}\varepsilon) > 0$ be a number with the property of the persistence for f. If $g \in E_\alpha(X)$ with $d_0(f, g) < \delta$, then for any

$x \in X$ there is $y \in X$ satisfying

$$d(f^n(x), g^n(y)) < \frac{1}{2}\varepsilon \quad \text{for every} \quad n \in \mathbb{Z}.$$

Define a map $k : X \to X$ by $k(x) = y$, where y is an element in X chosen by the property of the persistence for f as above. Then the map k is well-defined. In fact, let z be another element in X such that $d(f^n(x), g^n(z)) < \frac{1}{2}\varepsilon$ for all $n \in \mathbb{Z}$. Then we have $d(g^n(z), g^n(x)) < \varepsilon < \alpha$ for all $n \in \mathbb{Z}$. Hence we get $y = z$.

By the similar method used in [1],We can show that k in continuos. Sine we have choosed ε sufficiently small, We may regard that $k : X \to X$ is surjective. Further, k is injective. In fact, if $k(x) = k(y)$, then

$$\begin{aligned} d(f^n(x), f^n(y)) &\le d(f^n(x), g^n(k(x)) + d(g^n(k(x)), g^n(k(y)) \\ &\quad + d(g^n(k(y)), f^n(y)) < 2\varepsilon < e(f) \end{aligned}$$

and therefore $x = y$.

Let define $h : X \to X$ by $h(x) = k^{-1}(x)$. Then it is easy to show that $f \circ h(x) = h \circ g(x)$ for all $x \in X$. Using the similar method used in the proof of the continuity of k we can show that h is continuous and this completes the proof of the theorem.

Take and fix $\alpha > 0$. Then we can split $\overline{C(f)}$ into a union $\overline{C(f)} = \cup C_\lambda$ of equivalence classes C_λ under the α-relation in $\overline{C(f)}$.

LEMMA 2.4. *For $\alpha > 0$, every $x \in \overline{C(f)}$ is α-related to $f^k(x)$ in $\overline{C(f)}$ for all $k > 0$.*

PROOF. Let $x \in \overline{C(f)}$. Using the continuity of f we can take $\beta > 0$ with $\beta < \alpha$ such that $d(z, x) < \beta$ implies $\max\{d(f^i(x), f^i(z)) : 0 \le i \le k+1\} < \alpha$. Since $x \in \overline{C(f)}$ there is $z \in B(x, \beta) \cap C(f)$ and integer $\ell \ge k+2$ with $f^\ell(z) \in B(x, \beta)$. Then the sequence $\{f^k(x), f^{k+1}(z),$

$\cdots, f^{\ell-1}(z), x\}$ is an α-pseudo-orbit in $\overline{C(f)}$ from $f^k(x)$ to x. Obviously $\{x, f(x), \cdots, f^k(x)\}$ is α-pseudo-orbit from x to $f^k(x)$.

This lemma shows that each equivalence class C_λ is f-invariant.

LEMMA 2.5. *Each equivalence class C_λ is open and closed in $\overline{C(f)}$.*

PROOF. First, we show that each C_λ is closed in $\overline{C(f)}$. To see this, we choose a sequence $\{x_i\}$ in C_λ which converges to x'. Take $\beta > 0$ with $0 < \beta < \frac{1}{2}\alpha$ such that $f(B(x', \beta)) \subset B(f(x'), \frac{1}{2}\alpha)$. Let $s > 0$ be an integer with $d(x_s, x') < \beta$. Also, let $y \in B(x', \beta) \cap C(f)$ and $f^\ell(y) \in B(x', \beta)$ for some $\ell > 0$. Then the sequence $\{x', f(y), f^2(y), \cdots, f^{\ell-1}(y), x_s\}$ is an α-pseudo-orbit from x' to x_s in $\overline{C(f)}$. On the otherhand, take $\gamma > 0$ such that $B(x_s, \gamma) \subset B(x', \frac{1}{2}\alpha)$ and $f(B(x_s, \gamma)) \subset B(f(x_s), \frac{1}{2}\alpha)$. Let $w \in B(x_s, \gamma) \cap C(f)$ and $f^n(w) \in B(x_s, \gamma)$ for some $n > 0$. Then the sequence $\{x_s, f(w), f^2(w), \cdots, f^{n-1}(w), x'\}$ is an α-pseudo-orbit from x_s to x' in $\overline{C(f)}$. This implies that x_s is α-related to x' in $\overline{C(f)}$ and so $x' \in C_\lambda$. Hence C_λ is closed in $\overline{C(f)}$.

Next, we show that C_λ is open in $\overline{C(f)}$. Let $x \in C_\lambda$. for every $y \in C_\lambda$ there is an α-pseudo-orbit $\{x_0 = x, x_1, \cdots, x_p = y\}$ from x to y in $\overline{C(f)}$. Choose ξ with $0 < \xi < \frac{1}{3}\alpha$ such that $f(B(x_0, \xi)) \subset B(x_1, \alpha)$. Then for every $a \in B(x_0, \xi) \cap \overline{C(f)}$ the sequence $\{a, x_1, x_2, \cdots, x_p\}$ is an α-pseudo-orbit from a to y. On the otherhand, let $\{y_0 = y, y_1, \cdots, y_q = x\}$ be an α-pseudo-orbit from y to x in $\overline{C(f)}$. Since $x \in B(f(y_{q-1}), \alpha)$, we can choose a point z in $B(f(y_{q-1}), \alpha) \cap B(x, \xi) \cap C(f)$. Since $z \in B(x, \xi) \cap C(f)$ there is an integer $m > 0$ with $f^m(z) \in B(x, \xi)$. Therefore, for each a in $B(x, \xi) \cap \overline{C(f)}$ the sequence $\{y_0, y_1, \cdots, y_{q-1}, z, f(z), \cdots, f^{m-1}(z), a\}$ is an α-pseudo-orbit from y to a. This means that $B(x, \xi) \cap \overline{C(f)} \subset C_\lambda$. Hence C_λ is open in $\overline{C(f)}$, and so completes the proof.

LEMMA 2.6. *If $f \in H(X)$ has the semi-shadowing property, then for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every $g \in H(X)$ with $d_0(f,g) < \delta$ and for every periodic point x of g, there exists y in $C(f)$ satisfying $d(f^i(y), g^i(x)) < \varepsilon$ for every $i \in \mathbb{Z}$.*

PROOF. Let $\varepsilon > 0$ be given and let $\delta = \delta(\frac{1}{2}\varepsilon) > 0$ be the number with the property of the weak topological stability for f. For $g \in H(X)$ with $d_0(f,g) < \delta$ and $x \in X$ with $g^m(x) = x$ for some $m \geq 0$, consider $\{x_{mi+j}\} = \{g^j(x)\}$ for $i \in \mathbb{Z}$ and $0 \leq j < m$. Since f has the semi-shadowing property there exists z in X such that

$$(1) \qquad d(f^{mi+j}(z), g^{mi+j}(x)) < \frac{1}{2}\varepsilon$$

for $i \in \mathbb{Z}$ and $0 \leq j < m$. In particular, we have $f^{mi}(z) \in B(x, \frac{1}{2}\varepsilon)$ for all $i \in \mathbb{Z}$. Hence we have $\overline{O_{f^m}(z)} \subset \overline{B(x, \frac{1}{2}\varepsilon)}$. By the invariance of $\overline{O_{f^m}(z)}$ for f^m, there is a minimal set A of $\overline{O_{f^m}(z)}$ for f^m. By the minimality of A, for all $y \in A$, we have

$$\omega_{f^m}(y) = \alpha_{f^m}(y) = A \subset \overline{O_{f^m}(z)} \subset \overline{B(x, \frac{1}{2}\varepsilon)}.$$

Thus

$$y \in \omega_{f^m}(y) \cap \alpha_{f^m}(y) \subset \omega_f(y) \cap \alpha_f(y)$$

and so we conclude that $y \in C(f)$. It is sufficient to show that $d(f^{mi+j}(y), g^j(x)) < \varepsilon$ for every $i \in \mathbb{Z}$ and $0 \leq j < m$. To show this assume $d(f^{mi_0+j_0}(y), g^{j_0}(x)) \geq \varepsilon$ for some $i_0 \in \mathbb{Z}$ and $0 \leq j_0 < m$. Let $mi_0 + j_0 > 0$. Since $y \in \omega_{f^m}(y)$ we can shoose a sequence $\{f^{m\ell_i}(y)\}$ converges to y as $\ell_i \to +\infty$. Hence, by the continuity of f there is sufficiently large L in $\{\ell_i\}$ such that $d(f^{mL+j_0}(y), g^{j_0}(x)) \geq \varepsilon > \frac{1}{2}\varepsilon$. But this contradicts (1) and this completes the proof.

THEOREM 2.7. *Let $f \in H(X)$. If $f : \Omega(f) \to \Omega(f)$ has the semi-shadowing property, then so does $f : \overline{C(f)} \to \overline{C(f)}$.*

PROOF. Let $\varepsilon > 0$ be arbitrary. Then we can select $\delta_1 = \delta_1(\frac{1}{2}\varepsilon) > 0$ satisfying the property of the semi-shadowing property for $f \in H(\Omega(f))$ and the result of Lemma 2.4. Also, let

$\delta_2 = \delta_2(\delta_1) > 0$ with $\delta_2 < \delta_1$ as in Lemma 2.1. By Lemma 2.3, $\overline{C(f)}$ can be split into a finite union, $\overline{C(f)} = \cup_{i=1}^{k} C_i$, of equivalence class C_i under δ_1-relation. Let $\alpha = \min\{d(C_i, C_j) : 1 \le i, j \le k\}$ and take β with $0 < \beta < \min\{\alpha, \delta_1, \delta_2\}$ and let for $g \in H(\overline{C(f)}), d(f(z), g(z)) < \beta$ for all $z \in \overline{C(f)}$. For $x \in \overline{C(f)}$, consider a finite set

$$A_n = \{g^{-n}(x), g^{-n+1}(x), \cdots, g^n(x)\}.$$

Since $d(f(g^i(x)), g^{i+1}(x)) < \beta$ for $1 \le i \le n$. The set A_n is β-pseudo-orbit for f. Since each equivalence class C_i is f-invariant and $\beta < \alpha$ the β-pseudo-orbit $\{A_n\}$ is contained in some $C_j, 1 \le j \le k$. Therefore there exists a δ_1-pseudo-orbit $\{z_0^n, z_1^n, \cdots, z_{n_k}^n\}$ from $g^{-n}(x)$ to $g^n(x)$. Let define a set $\{(w_i, v_i)\}$ in $X \times X$ as follows

$$(w_i, v_i) = \begin{cases} (f(g^{i-1}(x)), g^i(x)), & 1 \le i \le n \\ (f(z_{i-n-1}^n), z_{i-n}^n), & n+1 \le i \le n+n_k \\ (f(g^{i-2n-n_k-1}(x), g^{i-2n-n_k}(x)), & n+n_k+1 \le i \le 2n+n_k. \end{cases}$$

Then the set $\{(w_i, v_i)\}_{i=1}^{2n+n_k}$ satisfies the hypothesis of Lemma 2.1. Thus there exists $a \in H(X)$ such that

$$d_0(a, 1_X) < \delta_1 \quad \text{and} \quad a(w_i) = v_i,$$

put $k = af$. Then we have $d_0(f(x), k(x)) < \delta$, and $k^m(x) = x, m = 2n + n_k$. Hence there is $y_n \in C(f)$ with $d(k^i(x), f^i(y_n)) < \dfrac{1}{2}\varepsilon$ for every $i \in \mathbb{Z}$. In particular, $d(f^i(y_n), g^i(x)) < \frac{1}{2}\varepsilon$ for $-n \le i \le n$. Let y_n converges to y as $n \to \infty$. Then it is easy to show that

$$d(f^i(y), g^i(x)) \le \frac{1}{2}\varepsilon < \varepsilon$$

for every $i \in \mathbb{Z}$. Hence the proof is complete.

Let $f \in H(X)$ and let $W^s_\varepsilon(x)$ and $W^u_\varepsilon(x)$ be the local stable and local unstable set of x for f. It was shown that if $\overline{\mathrm{Per}(f)}$, the closure of the set of periodic points of f, is hyperbolic set for f, then for $\varepsilon > 0$ sufficiently small there is $\delta > 0$ such that $x, y \in \overline{\mathrm{Per}(f)}$ and $d(x, y) < \delta$ implies

$$W^s_\varepsilon(x) \cap W^u_\varepsilon(y) = \{\text{one point}\} \subset \overline{\mathrm{Per}(f)}.$$

PROPOSITION 2.7. *Let $f \in H(X)$ be expansive and have the semi-shadowing property. Then, for sufficiently small $\varepsilon > 0$ there is $\delta > 0$ such that if $x, y \in C(f)$ with $d(x, y) < \delta$ then*

$$W^s_\varepsilon(x) \cap W^u_\varepsilon(y) = \{\textit{one point}\} \subset \overline{C(f)}.$$

PROOF. Let $e(f)$ be an expansive constant for f and $0 < \varepsilon < \frac{1}{3}e(f)$. Let $\delta_1 = \delta_1(\frac{1}{2}\varepsilon)$ with $0 < \delta_1 < \frac{1}{2}\varepsilon$ be a number satisfying the property of the semi-shadowing property for f and $\alpha = \alpha(\delta_1) < \frac{1}{2}\delta_1$ be the number as in Lemma 2.1. Choose $\delta > 0$ with $\delta < \frac{1}{2}\alpha$. Let $x, y \in C(f)$ with $d(x, y) < \delta$. Then we can choose sequences $\{n_i\}, \{m_i\}$ of positive integers satisfying :

$$n_i < n_j, m_i < m_j, \quad \text{if} \quad i < j; \quad \text{and}$$
$$d(x, f^{n_i}(x)) < \alpha, \quad d(y, f^{-m_i}(y)) < \alpha.$$

Take and fix an integer $k > 0$. Consider a set

$$A_k = \{x, f(x), \cdots, f^{n_k-1}(x), f^{-m_k}(y), f^{-m_k+1}(y), \cdots, f^{-1}(y), x\}.$$

Here, define the set $\{(p_i, q_i)\}_{i=0}^{n_k+m_k}$ in $X \times X$ by putting

$$(p_i, q_i) = \begin{cases} (f^i(x), f^i(x)), & 0 \le i \le n_k - 1 \\ (f^{n_k}(x), f^{-m_k}(y)), & i = n_k \\ (f^{-m_k+j}(y), f^{-m_k+j}(y)), & i = n_k + j,\ 1 \le j \le m_k - 1 \\ (y, x), & i = n_k + m_k. \end{cases}$$

Then the set $\{(p_i, q_i)\}$ satisfying the hypothesis of Lemma 2.1. Thus there exists $h_k \in H(X)$ such that

$$d_0(h_k, 1_X) < \delta_1 \quad \text{and} \quad h_k(p_i) = q_i$$

for $i = 0, 1, \cdots, n_k + m_k$. Put $g_k = h_k \circ f$. Then we have

$$d_0(f, g_k) < \delta_1 \quad \text{and} \quad g_k^{\ell_k}(x) = x,\ \ell_k = n_k + m_k - 1.$$

Hence there is z_k in $C(f)$ satisfying $d(g_k^i(x), f^i(z_k)) < \frac{1}{2}\varepsilon$ for all $i \in \mathbb{Z}$. Let $\{z_k\}$ converges to z as $k \to \infty$. Then $z \in \overline{C(f)} \cap W_\varepsilon^s(x) \cap W_\varepsilon^u(y)$. Using the expansivity of f, it is easy to show that $W_\varepsilon^s(x) \cap W_\varepsilon^u(y)$ is singleton and this completes the proof.

In the above result, we denote that

$$W_\varepsilon^s(x) \cap W_\varepsilon^u(y) = [x, y]$$

Then we can show that this bracket map $[\ ,\ ] : \Delta_\delta \overline{C(f)} \to X$ is continuous, where $\Delta_\delta \overline{C(f)}$ is the neighborhood of the diagonal in $\overline{C(f)} \times \overline{C(f)}$ defined by $\Delta_\delta \overline{C(f)} = \{(x, y) | x, y \in \overline{C(f)}, d(x, y) < \delta\}$. A closed invariant set is said to have the local product structure in Λ if for small ε and δ, $[x, y]$ belongs to Λ whenever $d(x, y) < \delta$, $x, y \in \Lambda$. Then by the above result , we obtain the following.

COROLLARY 2.8. *If $f \in H(X)$ is expansive and has the semi-shadowing property, then $\overline{C(f)}$ has a local product structure.*

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