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An Extended Version of Integral Continuitity Theorem

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ABSTRACT. The goal of this note is to give an extended version of integral continuity theorem, which may be used in the study of dynamics on noncompact spaces.

A flow on a topological space M is a continuous map $\phi : M \times \mathbb{R} \to M$ such that

(1) $\phi(x, 0) = x, x \in M$,
(2) $\phi(\phi(x, s), t) = \phi(x, s + t), x \in M, s, t \in \mathbb{R}$.

It follows easily that the transition maps $\phi_t : M \to M$ defined by $\phi_t(x) = \phi(x, t), t \in \mathbb{R}$, are homeomorphisms. The orbit of ϕ passing through a point $x \in M$ is given by the set $\{\phi_t(x) : t \in \mathbb{R}\}$.

Throughout the paper we let M denotes a locally compact metric space with metric d, ϕ a flow on M, and $\mathbb{R}^+$ a set of all positive real numbers.

One of the fundamental questions in the study of dynamics is: *How do orbits depend on their initial points?* For example, if the distance $d(x_0, y_0)$ between points x_0 and y_0 of M is small then what is the behavior of the distance between $\phi_t(x_0)$ and $\phi_t(y_0)$ as t gets large? For these questions, there is a nice answer which is well-known and called the "Integral Continuity Theorem ".

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THEOREM 1. *(Integeral Continuity Theorem). For any point* $p \in M$*, any number* $T > 0$ *and any constant* $\varepsilon > 0$*, there exists constant* $\delta = \delta(p, T, \varepsilon) > 0$ *such that if* $d(p, q) < \delta$ *for* $q \in M$*, then* $d(\phi_t(p), \phi_t(q)) < \varepsilon$ *for all* $t \in [0, T]$.

In many literatures, the dynamics (for example, chain recurrence, pseudo-orbit tracing property, topological stability, hyperbolicity, attractor, etc) in compact spaces are mainly considered, even if it is valuable to study such dynamics in noncompact spaces. Recently, Peixoto and Pugh generalized the concept of chain recurrence in noncompact space, using the continuous positive real valued functions on M, instead of the constants (for details, see [1]).

However we guess that the above theorem is not suitable for the generalization of dynamics in compact spaces to those in noncompact spaces. Here we give another version of "Intergral Continuity Theorem" which may be used in the study of dynamics in noncompact spaces.

THEOREM 2. *For any number* $T > 0$ *and any continuous map* $\varepsilon : M \to \mathbb{R}^+$*, there exists a continuous map* $\delta : M \to \mathbb{R}^+$ *such that if* $d(p, q) < \delta(p)$ *for* $q \in M$*, then* $d(\phi_t(p), \phi_t(q)) < \varepsilon(\phi_t(p))$ *for all* $t \in [0, T]$.

PROOF. For each $p \in M$, let $K = \{\phi_t(p) : 0 \leq t \leq T\}$. Since K is compact, there exists $a > 0$ with $inf\{\varepsilon(\phi_t(p)) : 0 \leq t \leq T\} = a$. By Theorem 1, there is $\alpha > 0$ such that $d(p, y) < \alpha$ implies $d(\phi_t(p), \phi_t(y)) < \frac{a}{2}$ for $0 \leq t \leq T$. Then we have

$$\overline{\phi_t(B_\alpha(p))} \subset \overline{B_{\frac{a}{2}}(\phi_t(p))} \subset B_a(\phi_t(p)) \subset B_{\varepsilon(\phi_t(p))}(\phi_t(p)),$$

for $0 \leq t \leq T$, where $B_t(p) = \{q \in M : d(p, q) < \varepsilon\}$. Let us define a map $h : M \to \mathbb{R}^+$ by

$$h(p) = \sup_{\alpha} \left\{ \begin{array}{ll} 0 < \alpha < \dfrac{1}{2} : \overline{B_\alpha(p)} \text{ is compact , and} & \\ \qquad \phi_t(\overline{B_\alpha(p)}) \subset B_{\varepsilon(\phi_t(p))}(\phi_t(p)) \text{ for } 0 \leq t \leq T \end{array} \right\}.$$

For any $0 < \alpha < h(p)$, choose $\beta > 0$ such that $\alpha < \beta \leq h(p)$ and $\phi_t(\overline{B_\beta(p)}) \subset B_{\varepsilon(\phi_t(p))}(\phi_t(p))$, for $0 \leq t \leq T$. Let $L = \{\phi_t(x) : x \in \overline{B_\beta(p)}, 0 \leq t \leq T\}$. Since $\overline{B_\beta(p)}$ and L are compact, we can choose a continuous map $\gamma : L \to \mathbb{R}^+$ such that

$$\phi_t(\overline{B_\beta(p)}) \subset B_{\varepsilon(\phi_t(p))}(\phi_t(p)) \textit{ and } \gamma(\phi_t(x)) < \varepsilon(\phi_t(x)),$$

for $t \in [0, T]$ and $\phi_t(x) \in L$. Using the compactness of L, select $\lambda > 0$ such that

$$\gamma(\phi_t(x)) \geq \lambda \textit{ and } (\varepsilon - \gamma)(\phi_t(x)) \geq \lambda$$

for each $\phi_t(x) \in L$. Since $\varepsilon : L \to \mathbb{R}^+$ is continuous, there exists $0 < \lambda_0 < \frac{\lambda}{2}$ such that $d(x, y) < \lambda_0$ implies $d(\varepsilon(x), \varepsilon(y)) < \frac{\lambda}{2}$. By Theorem 1, we can choose $\lambda_1 > 0$ such that $d(p, y) < \lambda_1$ implies $d(\phi_t(p), \phi_t(y)) < \lambda_0$ for $0 \leq t \leq T$. If y is a point in M with $d(p, y) < min(\lambda_1, \frac{1}{2}(\beta - \alpha))$, then we have

$$B_\alpha(y) \subset B_\beta(p) \textit{ and } B_{\gamma(\phi_t(p))}(\phi_t(p)) \subset B_{\varepsilon(\phi_t(y))}(\phi_t(y)),$$

for $0 \leq t \leq T$. In fact, if $d(\phi_t(p), l) < \gamma(\phi_t(p))$ then

$$\begin{aligned} d(\phi_t(y), l) &\leq d(\phi_t(y), \phi_t(p)) + d(\phi_t(p), l) \\ &< \frac{\lambda}{2} + \gamma(\phi_t(p)) \\ &< \varepsilon(\phi_t(p)) - \frac{\lambda}{2} < \varepsilon(\phi_t(y)), \end{aligned}$$

for $0 \le t \le T$. Hence we get

$$\phi_t(\overline{B_\alpha(y)}) \subset \phi_t(\overline{B_\beta(y)}) \subset B_{\varepsilon(\phi_t(p))}(\phi_t(p)) \subset B_{\varepsilon(\phi_t(y))}(\phi_t(y)),$$

for $0 \le t \le T$. This means that $\alpha \le h(y)$, and so the map h is lower semicontinuous.

Let $\wp = \{B_{\frac{h(p)}{2}}(p) : p \in M\}$ be an open covering of M. Since M is paracompact, there exists a locally finite open refinement $\aleph$ of $\wp$, say $\aleph = \{U_i : i \in \Lambda\}$. Let $\{f_i : M \to [0,1], i \in \Lambda\}$ be a partition of unity subordinated to $\aleph$. Then we have : $\{i \in \Lambda : f_i(x) \neq 0\}$ is finite , $\sum_{i \in \Lambda} f_i(x) = 1$ for each $x \in M$, and $\overline{\{x : f_i(x) \neq 0\}} \subset U_i$ for each $i \in \Lambda$. Since h is lower semicontinuous, for each $i \in \Lambda$, there exists $r_i > 0$ such that $h(\overline{(U_i)}) > 2r_i$. Define a continuous map $\delta : M \to \mathbb{R}^+$ by $\delta(x) = \sum_{i \in \Lambda} r_i f_i(x)$. If $f_i(x) \neq 0$ for $i \in \Lambda$, then $h(x) > 2r_i$. Hence we get

$$\delta(x) = \sum_i r_i f_i(x) \le \sum_i \frac{h(x)}{2} f_i(x) = \frac{h(x)}{2},$$

for each $x \in M$. By the definition of h, we have $\phi_t(B_{\delta(p)}(p)) \subset B_{\varepsilon(\phi_t(p))}(\phi_t(p))$ for $p \in M$ and $0 \le t \le T$, and so completes the proof.

REMARK 3. If the phase space M is compact then the new version of Integral Continuity Theorem is equivalent to the old one.

REFERENCES

1. C.Pugh, *The planar closing lemma for chain recurrence*, Trans. Amer. Math. Soc. **341** (1994), 173- -192.
2. K.S. Sibirsky, *Introduction to topological dynamics*, Noordhoff, Leyden,, Netherlands, 1975.

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