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k_ε-Stability in Differential Systems

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Abstract. We investigate some properties k_ε-stability which is an h-stability of exponential type.

1. Exponential asymptotic stability (EAS) and uniform Lipschitz stability (ULS) are the basic notions in stability theory for differential systems. EAS and ULS were investigated in [2] and [5] for ordinary differential equations. For functional differential equations EAS and ULS were studied in [4].

Pinto [6, 7, 8] introduced h-stability (hS) which is an important extension of the notions of EAS and ULS. He introduced the concept of hS with the intention of obtaining results about stability for a weakly stable systems (at least, weaker than those given by EAS and ULS).

In this paper we investigate some properties of k_ε-stability which is an h-stability of exponential type.

2. We consider the nonlinear nonautonomous differential system

$$x' = f(t,x), \qquad x(t_0) = x_0, \tag{1}$$

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where $f \in C^1(I_a \times D), I_a = [a, \infty)$ and D is a region of $\mathbb{R}^n$ containing the origin. We assume that the Jacobian matrix $f_x = \partial f/\partial x$ exists and is continuous on $I_a \times D$ and that $f(t,0) = 0$. The symbol $|\cdot|$ denotes arbitrary vector norm in $\mathbb{R}^n$.

Let $x(t) = x(t, t_0, x_0)$ be denoted by the unique solution of (1) through (t_0, x_0) for all $t \geq t_0 \geq a$ and for all $x_0 \in D$.

Also, we consider the associated variational system

$$z' = f_x(t, x(t, t_0, x_0))z, \qquad z(t_0) = z_0. \tag{2}$$

Let $\phi(t, t_0, x_0) = \frac{\partial}{\partial x_0} x(t, t_0, x_0)$ be the fundamental matrix solution of (2).

Pinto[4] introduced h-system by the following statements.

The system (1) (or the trivial solution $x = 0$ of (1)) is called *h-stable* (hS) if there exist $c \geq 1, \delta > 0$ and a positive bounded continuous function h on I_a such that

$$|x(t)| \leq c|x_0|h(t)h(t_0)^{-1} \tag{$*$}$$

for $t \geq t_0 \geq a$ and $|x_0| \leq \delta$,

Also, the system (2) is h-stable if there exist $c \geq 1, \delta > 0$ and a positive bounded continuous function such that

$$|\phi(t, t_0, x_0)| \leq ch(t)h(t_0)^{-1} \tag{$**$}$$

for $t \geq t_0 \geq a$ and $|x_0| \leq \delta$.

If, in $(*)$ and $(**)$, $h(t)$ is replaced by

$$h_\varepsilon(t) = h(t)e^{c\varepsilon t}, \quad \varepsilon > 0,$$

then the system is called h_ε*-stable* , and $h(t)$ is replaced by

$$k_\varepsilon(t) = h(t)e^{c\int_{t_0}^{t}(\lambda(s)+\varepsilon)\, ds}, \quad \lambda \in L_1(I_a)$$

then the system is called *k_ε-stable.*

We need the Alekseev formula for a comparison between the solutions of (1) and the solutions of the perturbed nonlinear system

$$y' = f(t,y) + g(t,y), \quad y(t_0) = y_0, \tag{3}$$

where $g \in C(I_a \times D, \mathbb{R}^n)$. We let $y(t) = y(t,t_0,y_0)$ denote the solution of (3) passing through the point (t_0, y_0) in $I_a \times D$.

LEMMA 1 [1]. *If $y_0 \in D \subset \mathbb{R}^n$, then for all t such that $x(t,t_0,y_0) \in D \subset \mathbb{R}^n$,*

$$y(t,t_0,y_0) = x(t,t_0,y_0) \; + \int_{t_0}^{t} \phi(t,s,y(s))g(s,y(s))\, ds.$$

LEMMA 2 [5]. *Assume that $x(t,t_0,x_0)$ and $x(t,t_0,y_0)$ are solutions of (1) through (t_0,x_0) and (t_0,y_0), respectively, which exist for $t \geq t_0$ and such that x_0 and y_0 belong to a convex subset $\hat{D}$ of $\mathbb{R}^n$. Then for $t \geq t_0$,*

$$x(t,t_0,y_0) - x(t,t_0,x_0) = \int_0^1 \phi(t,t_0,x_0 + s(y_0 - x_0))ds \cdot (y_0 - x_0).$$

3. Note that if $z = 0$ of (2) is hS, then $x = 0$ of (1) is also hS because

$$x(t,t_0,x_0) = (\int_0^1 \phi(t,t_0,sx_0)\, ds)x_0,$$

by Lemma 2 .

For the converse, we need a condition

$$|f_x(t,x) - f_x(t,0)| \leq v(t)|x| \tag{4}$$

for x in a neighborhood of the origin, where $\int_{t_0}^{\infty} v(s)\, ds < \infty$.

LEMMA 3 [3]. *Under the condition (4), if $x = 0$ of (1) is hS, then $z = 0$ of (2) is hS.*

THEOREM 1. *If $z = 0$ of (2) is hS and $g(t, y) = g_1(t, y)$ is satisfied by*

$$|g_1(t, y)| \le \lambda(t)|y|, \qquad \lambda \in C(I_a), \tag{5}$$

then the perturbed system (3) is also hS when $\lambda \in L_1(I_a)$.

PROOF. See [6].

Consider the perturbation $g(t, y) = g_2(t, y)$ with the conditions

(6)

$$|g_2(t, y)| \le \nu(t), \Lambda(t) = \int_t^{t+1} \nu(s)\, ds \to 0 \quad \text{as} \quad t \to \infty, \quad \nu \in C(I_a).$$

LEMMA 4. *Under the conditions (6), assume that the positive continuous function h defined on I_a satisfy the following conditions:*

$$\limsup_{t\to\infty} h(t) \int_a^t h(s)^{-1}\, ds = M \qquad \text{and} \tag{7}$$

(8)

$$0 < \liminf_{t\to\infty} h(t) \int_t^{t+1} h(s)^{-1}\, ds \le \limsup_{t\to\infty} h(t) \int_t^{t+1} h(s)^{-1}\, ds < \infty.$$

Then $\lim_{t\to\infty} h(t) \int_T^t h(s)^{-1} \nu(s)\, ds = 0$ for all $T \ge a$.

PROOF. See Lemmas 1 and 2 of [6].

LEMMA 5[6]. *If h satisfies the condition (7), then there is a positive constant N such that*

$$h(t) \le N e^{-\frac{t}{M}}$$

for $t \ge a$ and M as in (7).

THEOREM 2. *Assume that the system (2) is hS and $g(t,y)$ satisfy (6), where h satisfies (7) and (8). Then the zero solution of (3) tends to zero as $t \to \infty$.*

PROOF. See [Theorem 6, 2].

Now, we obtain an h_ε-stability property resulting from the perturbation of an h-system.

THEOREM 3. *Assume that the system (2) is hS and $g(t,y) = g_3(t,y)$ satisfy*

(9) $$|g_3(t,y)| \leq \varepsilon |y|, \qquad \varepsilon > 0,$$

for y in a neighborhood of the origin, uniformly in t. Then the perturbed system of (3) is h_ε-stable, where $h_\varepsilon = h(t)e^{c\varepsilon t}$, and all solutions of (3) tend to zero as $t \to \infty$ if h satisfy (7) and $c\varepsilon < \frac{1}{M}$.

PROOF. Using Alekseev's formula, we obtain

$$y(t) = y(t,t_0,y_0) = x(t,t_0,y_0) + \int_{t_0}^{t} \phi(t,s,y(s))g_3(s,y(s))\, ds.$$

Therefore we have

$$\begin{aligned} |y(t)| = |y(t,t_0,y_0)| &\leq |x(t,t_0,y_0)| + \int_{t_0}^{t} |\phi(t,s,y(s))||g_3(s,y(s))|\, ds \\ &\leq c|y_0|h(t)h(t_0)^{-1} + c\int_{t_0}^{t} h(t)h(s)^{-1}\varepsilon|y(s)|\, ds. \end{aligned}$$

By Gronwall's inequality, we obtain

$$\begin{aligned} |y(t)| &\leq c|y_0|h(t)h(t_0)^{-1} \exp(c\int_{t_0}^{t} \varepsilon\, ds) \\ &= c|y_0|h(t)h(t_0)^{-1}e^{c\varepsilon(t-t_0)} \\ &= c|y_0|h_\varepsilon(t)h_\varepsilon(t_0)^{-1}, \end{aligned}$$

where $h_\varepsilon(t) = h(t)e^{c\varepsilon t}$.

If h satisfies (7), there exists $N > 0$ such that $|h(t)| \leq Ne^{-\frac{t}{M}}$. Thus $|h_\varepsilon(t)| \leq Ne^{(c\varepsilon - \frac{1}{M})t}$. Therefore if $c\varepsilon < \frac{1}{M}$, then $y = 0$ of (3) is h_ε-stable and $h_\varepsilon(t) \to 0$ as $t \to \infty$. This completes the proof.

Next, we investigate k_ε-stability for the perturbed system

$$y' = f(t,y) + g_1(t,y) + g_3(t,y), \qquad y(t_0) = y_0. \tag{10}$$

THEOREM 4. *Under the conditions (5), (7) and (9), suppose that the system (2) is hS. Then*

(i) *for all $t_0 \geq a$ and y_0 small enough, all solutions $y(t,t_0,y_0)$ of (10) are defined for all $t \geq t_0$,*

(ii) *the system (10) is a k_ε-stable and all solutions $y(t,t_0,y_0)$ of (10) tend to zero as $t \to \infty$ if $\lambda \in L_1(I_a)$ and $c\varepsilon < \frac{1}{M}$, where $k_\varepsilon(t) = h(t)\exp\{c\int_{t_0}^t (\lambda(s) + \varepsilon)\, ds\}$.*

PROOF. By Alekseev's formula, we have

$$y(t) = x(t,t_0,y_0) + \int_{t_0}^t \phi(t,s,y(s))\{g_1(s,y(s)) + g_3(s,y(s))\}\, ds.$$

Thus

$$|y(t)| \leq c|y_0|h(t)h(t_0)^{-1} + ch(t)\int_{t_0}^t h(s)^{-1}(\lambda(s)+\varepsilon)|y(s)|\, ds$$

or

$$h(t)^{-1}|y(t)| \leq c|y_0|h(t_0)^{-1} + c\int_{t_0}^t h(s)^{-1}(\lambda(s)+\varepsilon)|y(s)|\, ds.$$

Then by Gronwall's inequality,

$$|y(t) \le c|y_0|h(t)h(t_0)^{-1}\exp\{c\int_{t_0}^{t}(\lambda(s)+\varepsilon)\,ds\}.$$

Therefore the solutions $y(t,t_0,y_0)$ of (10) are defined for all $t \ge t_0$. Put $k_\varepsilon(t) = h(t)\exp\{c\int_a^t(\lambda(s)+\varepsilon)\,ds\}$. Then $|y(t)| \le c|y_0|k_\varepsilon(t)k_\varepsilon(t_0)^{-1}$. By Lemma 5, $h(t) \le Ne^{-\frac{t}{M}}$, $N > 0$ a constant. Then

$$\begin{aligned} k_\varepsilon(t) &= h(t)\exp\{c\int_a^t(\lambda(s)+\varepsilon)\,ds\} \\ &\le Ne^{-\frac{t}{M}}\cdot Le^{c\varepsilon(t-a)} \\ &= c_1 e^{(c\varepsilon-\frac{1}{M})t}, \qquad \text{where} \quad c_1 = NLe^{-ca\varepsilon}. \end{aligned}$$

If $c\varepsilon < \frac{1}{M}$, then $k_\varepsilon(t)$ is bounded and the system (10) is k_ε-stable. Also, $k_\varepsilon(t) \to 0$ as $t \to \infty$. Therefore $y(t)$ tend to zero as $t \to \infty$. This completes the proof.

REMARK 1. In Theorem 4, we can prove the system (10) as a perturbed system of (3-1) $y' = f(t,y) + g_1(t,y)$ under the following condition:

$$\text{(H)} \qquad |G_x(t,x) - G_x(t,0)| \le K(t)|x|, \quad K \in L_1(I_a),$$

where $G(t,x) = f(t,x) + g_1(t,x)$. For, by Theorem 1, the system (3-1) is hS . So the fundamental matrix solution of (3-1) is hS by (H) and Lemma 3. Therefore for the solution of (10) we have the same conclusion of Theorem 3.

LEMMA 6. *Under the hypotheses (6), (7) and (8),*

$$k_\varepsilon(t)\int_{t_0}^{t}k_\varepsilon(s)^{-1}\nu(s)\,ds$$

tend to zero as $t \to \infty$, *where* $k_\varepsilon(t) = h(t)\exp\{c\int_a^t(\lambda(s)+\varepsilon)\ ds\}$, $\lambda \in L_1(I_a)$ *and* $c\varepsilon < \frac{1}{M}$.

PROOF. Since $\Lambda(t) \to 0$ as $t \to \infty$, for every $\delta > 0$ there exists T such that $\Lambda(t) < \delta$ for $t \geq T$. Therefore we can compute

$$\begin{aligned}
&k_\varepsilon(t)\int_{t_0}^t k_\varepsilon(s)^{-1}\nu(s)\ ds \\
=&k_\varepsilon(t)\int_{t_0}^T k_\varepsilon(s)^{-1}\nu(s)\ ds + k_\varepsilon(t)\int_T^t k_\varepsilon(s)^{-1}\nu(s)\ ds.
\end{aligned}$$

The first term of the right side tends to zero by the proof of Theorem 4 and the second term also tends to zero because

$$\begin{aligned}
k_\varepsilon(t)\int_T^t k_\varepsilon(s)^{-1}\nu(s)\ ds &= h(t)\exp\{c\int_a^t(\lambda(\tau)+\varepsilon)\ d\tau\} \\
&\quad \cdot \int_T^t h(s)^{-1}\exp\{c\int_s^a(\lambda(\tau)+\varepsilon)\ d\tau\}\ \nu(s)\ ds \\
&= h(t)\int_T^t h(s)^{-1}\nu(s)\exp\{c\int_s^t(\lambda(\tau)+\varepsilon)\ d\tau\}\ ds \\
&\leq Lh(t)\int_T^t h(s)^{-1}\nu(s)\ ds,
\end{aligned}$$

where $L = \sup_{T\leq s\leq t}\exp\{c\int_s^t(\lambda(\tau)+\varepsilon)\ d\tau\}$. By Lemma 4 , $h(t)\int_{t_0}^t h(s)^{-1}\nu(s)\ ds$ tends to zero as $t \to \infty$. This completes the proof.

THEOREM 5. *Under the conditions (5), (6), (7), (8), and (9), assume that the system (2) is hS and*

(11) $$|F_x(t,x) - F_x(t,0)| \leq K(t)|x|, \qquad K \in L_1(I_a),$$

where $F(t,x) = f(t,x) + g_1(t,x) + g_3(t,x)$. *Then for any* $t_0 \geq a$ *and* y_0 *small enough, the solutions* $y(t,t_0,y_0)$ *of the perturbed system*

(12) $$y' = f(t,y) + g_1(t,y) + g_2(t,y) + g_3(t,y)$$

are defined for $t \geq t_0$ and for all t_0 big enough, the solutions $y(t, t_0, y_0)$ tend to zero as $t \to \infty$ if $\lambda \in L_1(I_a)$ and $c\varepsilon < \frac{1}{M}$.

PROOF. Let

$$\begin{aligned} y' &= f(t,y) + g_1(t,y) + g_2(t,y) + g_3(t,y) \\ &= F(t,y) + g_2(t,y). \end{aligned}$$

From Theorem 4, the system $x' = F(t,x)$ is k_ε-stable. The variational system $z' = F_x(t, x(t, t_0, x_0))z$ is k_ε-stable, that is,

$$|\Phi(t, t_0, x_0)| \leq c k_\varepsilon(t) k_\varepsilon(t_0)^{-1}$$

where $\Phi(t, t_0, x_0) = \frac{\partial}{\partial x_0} x(t, t_0, x_0)$ is the fundamental matrix solution of the variational system $z' = F_x(t, x(t, t_0, x_0))z$ of (10), by (11) and Lemma 3. Therefore we have

$$y(t) = y(t, t_0, y_0) = x(t, t_0, y_0) + \int_{t_0}^{t} \Phi(t, s, y(s)) g_2(s, y(s))\, ds.$$

Thus

$$|y(t)| \leq c|y_0| k_\varepsilon(t) k_\varepsilon(t_0)^{-1} + c k_\varepsilon(t) \int_{t_0}^{t} k_\varepsilon(s)^{-1} \nu(s)\, ds.$$

Hence $y(t)$ is defined on the whole interval $[t_0, \infty)$. For t_0 big enough, the first term of the above tends to zero as $t \to \infty$ by Theorem 4 and the second term of the above tends to zero as $t \to \infty$ by Lemma 6.

REMARK 2. In Theorems 1 $\sim$ 5, if we assume that (4) and $x = 0$ of (1) is hS instead of the system (2) is hS, then we have the same conclusion by Lemma 3. In Theorem 5, if $\varepsilon = 0$, i.e., $g_3 = 0$ and $\lambda \in L_1(I_a)$, then the solutions of the system $y' = f(t,y) + g_1(t,y) + g_2(t,y)$ tend to zero as $t \to \infty$. Similarly, if $\lambda = 0$, i.e., $g_1 = 0$, then the zero solution of $y' = f(t,y) + g_2(t,y) + g_3(t,y)$ tends to zero as $t \to \infty$.

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