JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume 7, June 1994

Derivations on Semiprime Rings and Banach Algebras, I

*BYUNG-DO KIM AND **YANG-HI LEE

ABSTRACT. The aim of this paper is to give the partial answer of Vukman's conjecture [2]. From the partial answer we also generalize a classical result of Posner. We prove the following result: Let R be a prime ring with char(R) $\neq 2, 3,$ and 5. Suppose there exists a nonzero derivation $D : R \longrightarrow R$ such that the mapping $x \mapsto [[[Dx, x], x], x]$ is centralizing on R. Then R is commutative. Using this result and some results of Sinclair and Johnson, we generalize Yood's noncommutative extension of the Singer-Wermer theorem.

1. Introduction

Throughout, R represents an associative ring with center $Z(R)$. We shall write char(R) for the characteristic of ring R. We write $[x, y]$ for $xy - yx$. Recall that R is prime if $aRb = (0)$ implies that either $a = 0$ or $b = 0$, and is semiprime if $aRa = (0)$ implies $a = 0$. An additive mapping D from R to R is called a derivation if $D(xy) = (Dx)y + xDy$ holds for all $x, y \in R$. An additive mapping D from R to R is called a Jordan derivation if $D(x^2) = (Dx)x + xDx$ holds for all $x \in R$. A mapping F from R to R is said to be commuting on R if $[F(x), x] = 0$ holds for all $x \in R$, and is said to be centralizing on R if $[F(x), x] \in Z(R)$ holds for all $x \in R$. In 1990, J. Vukman [2] proved that in case there exists a nonzero derivation $D : R \longrightarrow R$, where R is a prime ring of characteristic different from 2 , 3, such that the mapping $x \mapsto [[Dx, x], x]$ is centralizing on R, R is commutative.

Received by the editors on June 30, 1994.
1980 *Mathematics subject classifications*: Primary 16N60 .

The main purpose of this paper is in solving Vukman's conjecture when $n = 4$. Hence we also generalize Theorem 2 and Theorem 3 in [3] in the same fashion.

2. Main Results

The following result is the partial answer of Vukman's conjecture [2].

THEOREM 2.1. *Let R be a noncommutative prime ring with char$(R) \neq 2$, 3, and 5. Suppose there exists a derivation $D : R \longrightarrow R$ such that the mapping $x \longmapsto [[[Dx, x], x], x]$ is centralizing on R. Then we have $D = 0$.*

PROOF. We define a mapping $B(\cdot,\cdot) : R \times R \longrightarrow R$ by

$$B(x,y) = [Dx, y] + [Dy, x], \quad x, y \in R.$$

Then, $B(\cdot,\cdot)$ is symmetric and additive in both arguments. A calculation shows that the relation

$$B(xy, z) = B(x, z)y + xB(y, z) + (Dx)[y, z] + [x, z]Dy. \tag{1}$$

holds for all $x, y, z \in R$. Now, let $e(x) = 2Dx$ for all $x \in R$. And, we also introduce a mapping f from R to R by $f(x) = B(x, x)$. We have

$$f(x) = [e(x), x], \quad x \in R. \tag{2}$$

Obviously, the mapping f satisfies the relation

$$f(x + y) = f(x) + f(y) + 2B(x, y), \quad x, y \in R. \tag{3}$$

Throughout the proof, we use the mapping $B(\cdot,\cdot)$ and the relations (1), (2), and (3) without specific reference. The assumption of the theorem can now be written in the form

$$[[[f(x), x], x], x] \in Z(R), \quad x \in R. \tag{4}$$

Conveniently, we will introduce the following arguments:

Let F, $G : R \longrightarrow R$ and $H : R \times R \longrightarrow R$ be the mappings such that $H(x, xy) = xH(x,y)$, $H(x, yx) = H(x,y)x$,

(a) $$F(x)H(x,y)G(x) = 0, \quad x, y \in R.$$

(LA):

Substituting xy for y in (a), one obtains

(b) $$F(x)xH(x,y)G(x) = 0, \quad x, y \in R.$$

And, left multiplication of (a) by x gives

(c) $$xF(x)H(x,y)G(x) = 0, \quad x, y \in R.$$

And so, subtracting (c) from (b), we have

$$[F(x), x]H(x,y)G(x) = 0, \quad x, y \in R.$$

By the similar method,

(RA):

$$F(x)H(x,y)[G(x), x] = 0, \quad x, y \in R.$$

First we intend to prove that the mappings $x \mapsto [[f(x), x], x]$ is commuting on R. In other words, we are going to prove that

$$[[[f(x), x], x], x] = 0, \quad x \in R.$$

The linearization of (4) gives

$$\begin{aligned}&[[[f(y),x],x],x]+2[[[B(x,y),x],x],x]+[[[f(x),x],x],y]\\&+[[[f(y),x],x],y]+2[[[B(x,y),x],x],y]+[[[f(x),x],y],x]\\&+[[[f(y),x],y],x]+2[[[B(x,y),x],y],x]+[[[f(x),x],y],y]\\&+[[[f(y),x],y],y]+2[[[B(x,y),x],y],y]+[[[f(x),y],x],x]\\&+[[[f(y),y],x],x]+2[[[B(x,y),y],x],x]+[[[f(x),y],x],y]\\&+[[[f(y),y],x],y]+2[[[B(x,y),y],x],y]+[[[f(x),y],y],x]\\&+[[[f(y),y],y],x]+2[[[B(x,y),y],y],x]+[[[f(x),y],y],y]\\&+2[[[B(x,y),y],y],y]\in Z(R),\quad x,y\in R.\end{aligned}$$

Replacing $-x$ for x in the above relation we have the new relation: and comparing the new relation with the above relation, we obtain

$$\begin{aligned}&[[[f(x),x],x],y]+[[[f(x),x],y],x]+[[[f(x),y],x],x]\\&+[[[f(x),y],y],y]+[[[f(y),x],x],y]+[[[f(y),x],y],x]\\&+[[[f(y),y],x],x]+2[[[B(x,y),x],x],x]+2[[[B(x,y),x],y],y]\\&+2[[[B(x,y),y],x],y]+2[[[B(x,y),y],y],x]\ \in Z(R),\quad x,y\in R.\end{aligned}\tag{5}$$

Substituting $2x$ for x in (5) and then using the fact that char$(R)\neq 2$, we arrive at

$$\begin{aligned}&4[[[f(x),x],x],y]+4[[[f(x),x],y],x]+4[[[f(x),y],x],x]\\&+[[[f(x),y],y],y]+[[[f(y),x],x],y]+[[[f(y),x],y],x]\\&+[[[f(y),y],x],x]+8[[[B(x,y),x],x],x]+2[[[B(x,y),x],y],y]\\&+2[[[B(x,y),y],x],y]+2[[[B(x,y),y],y],x]\ \in Z(R),\quad x,y\in R.\end{aligned}\tag{6}$$

And, subtracting (5) from (6), we get

$$\begin{aligned}&3[[[f(x),x],x],y]+3[[[f(x),x],y],x]+3[[[f(x),y],x],x]\\&+6[[[B(x,y),x],x],x]\in Z(R),\quad x,y\in R.\end{aligned}$$

But since $\mathrm{char}(R) \neq 3$, we have

$$[[[f(x), x], x], y] + [[[f(x), x], y], x] + [[[f(x), y], x], x] + 2[[[B(x, y), x], x], x] \in Z(R), \quad x, y \in R. \tag{7}$$

Replacing x^2 for y in (7), one obtains

$$5[[[f(x), x], x], x]x + 5x[[[f(x), x], x], x] \in Z(R), \quad x \in R, \tag{8}$$

and also, since $\mathrm{char}(R) \neq 5$, it follows from (8) that

$$[[[f(x), x], x], x]x + x[[[f(x), x], x], x] \in Z(R), \quad x \in R. \tag{9}$$

But since $[[[f(x), x], x], x] \in Z(R), x \in R$ and $\mathrm{char}(R) \neq 2$, we obtain from (9) that

$$[[[f(x), x], x], x]x \in Z(R), \quad x \in R \tag{10}$$

And so, it follows from (4) and (10) that

$$[[[f(x), x], x], x][x, y] = 0, \quad x, y \in R. \tag{11}$$

Now, substituting yz for y in (11), one obtains

$$[[[f(x), x], x], x]y[z, x] = 0, \quad x, y, z \in R. \tag{12}$$

And then, putting $[[f(x), x], x]$ for z in (12), we have

$$[[[f(x), x], x], x]y[[[f(x), x], x], x] = 0, \quad x, y \in R. \tag{13}$$

Therefore, by semiprimeness of R it is immediate from (13) that

$$[[[f(x), x], x], x] = 0, \quad x \in R. \tag{14}$$

From the linearization of (14), and using the assumption that char$(R) \neq 2, 3$, and 5, we arrive at

$$[[[f(x), x], x], y] + [[[f(x), x], y], x] + [[[f(x), y], x], x] + 2[[[B(x, y), x], x], x] = 0 \quad x, y \in R, \tag{15}$$

in the same fashion that makes it possible to obtain (7) from (4).

On the other hand, substituting xy for y in (15) we have

$$10[[f(x), x], x][y, x] + 10[f(x), x][[y, x], x] + 5f(x)[[[y, x], x], x] + e(x)[[[[y, x], x], x], x] = 0, \quad x, y \in R. \tag{16}$$

And also, replacing yx for y in (15) we arrive at

$$10[y, x][[f(x), x], x] + 10[[y, x], x][f(x), x] + 5[[[y, x], x], x]f(x) + [[[[y, x], x], x], x]e(x) = 0, \quad x, y \in R. \tag{17}$$

Subtracting $e(x)\times(17)$ from $(16)\times e(x)$, and using the condition that char$(R) \neq 5$ we obtain

$$2[[f(x), x], x][y, x]e(x) - 2e(x)[y, x][[f(x), x], x] + 2[f(x), x][[y, x], x]e(x) - 2e(x)[[y, x], x][f(x), x] + f(x)[[[y, x], x], x]e(x) - e(x)[[[y, x], x], x]f(x) = 0, \quad x, y \in R. \tag{18}$$

Now, applying (RA) to (18), and again doing (RA) to the relation so obtained, we get

$$2[[f(x), x], x][y, x][f(x), x] + 2[f(x), x][[y, x], x][f(x), x] + f(x)[[[y, x], x], x][f(x), x] - e(x)[[[y, x], x], x][[f(x), x], x] = 0, \quad x, y \in R. \tag{19}$$

And again, applying (LA) to (18), and doing (LA) to the relation so obtained, one obtains

$$\begin{aligned} &-2[f(x),x][y,x][[f(x),x],x]-2[f(x),x][[y,x],x][f(x),x]\\ &+[[f(x),x],x][[[y,x],x],x]e(x)-[f(x),x][[[y,x],x],x]f(x)\\ &=0, \quad x,y\in R. \end{aligned} \tag{20}$$

But, replacing yz for y in (16) it follows that

$$\begin{aligned} &10[[f(x),x],x]y[z,x]+10[f(x),x]y[[z,x],x]+20[f(x),x][y,x][z,x]\\ &+5f(x)y[[[z,x],x],x]+15f(x)[y,x][[z,x],x]+15f(x)[[y,x],x][z,x]\\ &+e(x)y[[[[z,x],x],x],x]+4e(x)[y,x][[[z,x],x],x]\\ &+6e(x)[[y,x],x][[z,x],x]+4e(x)[[[y,x],x],x][z,x]\\ &=0, \quad x,y,z\in R. \end{aligned} \tag{21}$$

Moreover, substituting $[f(x),x]$ for z in (21) we have

$$\begin{aligned} &10[[f(x),x],x]y[[f(x),x],x]+20[f(x),x][y,x][[f(x),x],x]\\ &+15f(x)[[y,x],x][[f(x),x],x]+4e(x)[[[y,x],x],x][[f(x),x],x]\\ &=0, \quad x,y\in R. \end{aligned} \tag{22}$$

Putting $e(x)$ instead of z in (21), after some calculations one obtains

$$\begin{aligned} &10[[f(x),x],x]yf(x)+10[f(x),x]y[f(x),x]+20[f(x),x][y,x]f(x)\\ &+5f(x)y[[f(x),x],x]+15f(x)[y,x][f(x),x]+15f(x)[[y,x],x]f(x)\\ &+4e(x)[y,x][[f(x),x],x]+6e(x)[[y,x],x][f(x),x]+4e(x)[[[y,x],x],x]f(x)\\ &=0, \quad x,y\in R. \end{aligned} \tag{23}$$

Let us write zy instead of y in (17). Then we have

$$\begin{aligned}
&10[z,x]y[[f(x),x],x] + 10[[z,x],x]y[f(x),x] + 20[z,x][y,x][f(x),x] \\
&+ 5[[[z,x],x],x]yf(x) + 15[[z,x],x][y,x]f(x) + 15[z,x][[y,x],x]f(x) \\
&+ [[[[z,x],x],x],x]ye(x) + 4[[[z,x],x],x][y,x]e(x) \\
&6[[z,x],x][[y,x],x]e(x) + 4[z,x][[[y,x],x],x]e(x) \\
&= 0, \quad x,y,z \in R. \qquad (24)
\end{aligned}$$

And also, substituting $e(x)$ for z in (24) we get

$$\begin{aligned}
&10f(x)y[[f(x),x],x] + 10[f(x),x]y[f(x),x] + 20f(x)[y,x][f(x),x] \\
&+ 5[[f(x),x],x]yf(x) + 15[f(x),x][y,x]f(x) + 15f(x)[[y,x],x]f(x) \\
&+ 4[[f(x),x],x][y,x]e(x) + 6[f(x),x][[y,x],x]e(x) \\
&+ 4f(x)[[[y,x],x],x]e(x) = 0, \quad x,y \in R. \qquad (25)
\end{aligned}$$

Hence, taking (25) from (23), we obtain

$$\begin{aligned}
&5[[f(x),x],x]yf(x) - 5f(x)[y,x][f(x),x] + 5[f(x),x][y,x]f(x) \\
&- 5f(x)y[[f(x),x],x] + 4e(x)[y,x][[f(x),x],x] - 4[[f(x),x],x][y,x]e(x) \\
&+ 6e(x)[[y,x],x][f(x),x] - 6[f(x),x][[y,x],x]e(x) \\
&+ 4e(x)[[[y,x],x],x]f(x) - 4f(x)[[[y,x],x],x]e(x) = 0, \quad x,y \in R. \qquad (26)
\end{aligned}$$

Applying (RA) to (26), we arrive at

$$\begin{aligned}
&5[[f(x),x],x]y[f(x),x] + 5[f(x),x][y,x][f(x),x] \\
&- 5f(x)[y,x][[f(x),x],x] - 4[[f(x),x],x][y,x]f(x) \\
&+ 6e(x)[[y,x],x][[f(x),x],x] - 6[f(x),x][[y,x],x]f(x) \\
&- 4f(x)[[[y,x],x],x]f(x) + 4e(x)[[[y,x],x],x][f(x),x] \\
&= 0, \quad x,y \in R. \qquad (27)
\end{aligned}$$

Similarly, applying (LA) to (26) one obtains

$$\begin{aligned}&5[[f(x),x],x][y,x]f(x)-5[f(x),x][y,x][f(x),x]-5[f(x),x]y[[f(x),x],x]\\&+4f(x)[y,x][[f(x),x],x]+6f(x)[[y,x],x][f(x),x]\\&-6[[f(x),x],x][[y,x],x]e(x)+4f(x)[[[y,x],x],x]f(x)\\&-4[f(x),x][[[y,x],x],x]e(x)=0,\quad x,y\in R.\end{aligned}\tag{28}$$

Applying (RA) to (27), we get

$$\begin{aligned}&5[[f(x),x],x]y[[f(x),x],x]+5[f(x),x][y,x][[f(x),x],x]\\&-4[[f(x),x],x][y,x][f(x),x]-6[f(x),x][[y,x],x][f(x),x]\\&-4f(x)[[[y,x],x],x][f(x),x]+4e(x)[[[y,x],x],x][[f(x),x],x]\\&=0,\quad x,y\in R.\end{aligned}\tag{29}$$

But then, since $4e(x)[[[y,x],x],x][[f(x),x],x] = -10[[f(x),x],x]y[[f(x),x],x] -20[f(x),x][y,x][[f(x),x],x]-15f(x)[[y,x],x][[f(x),x],x]$ for all $x,y \in R$ from (22), we obtain from (29) that

$$\begin{aligned}&5[[f(x),x],x]y[[f(x),x],x]+15[f(x),x][y,x][[f(x),x],x]\\&+4[[f(x),x],x][y,x][f(x),x]+6[f(x),x][[y,x],x][f(x),x]\\&+15f(x)[[y,x],x][[f(x),x],x]+4f(x)[[[y,x],x],x][f(x),x]\\&=0,\quad x,y\in R.\end{aligned}\tag{30}$$

On the other hand, applying (LA) to (28) we have

$$\begin{aligned}&-5[[f(x),x],x][y,x][f(x),x]-5[[f(x),x],x]y[[f(x),x],x]\\&+4[f(x),x][y,x][[f(x),x],x]+6[f(x),x][[y,x],x][f(x),x]\\&+4[f(x),x][[[y,x],x],x]f(x)-4[[f(x),x],x][[[y,x],x],x]e(x)\\&=0,\quad x,y\in R.\end{aligned}\tag{31}$$

Substituting $[f(x), x]$ for z in (24), we arrive at

$$\begin{aligned}&10[[f(x),x],x]y[[f(x),x],x]+20[[f(x),x],x][y,x][f(x),x]\\&+15[[f(x),x],x][[y,x],x]f(x)+4[[f(x),x],x][[[y,x],x],x]e(x)\\&=0,\quad x,y\in R.\end{aligned}\tag{32}$$

But, since $4[[f(x),x],x][[[y,x],x],x]e(x) = -10[[f(x),x],x]$ $y[[f(x),x],x]-20[[f(x),x],x][y,x][f(x),x]-15[[f(x),x],x][[y,x],x]f(x)$ holds for all $x, y \in R$ from (32), we obtain from (31) that

$$\begin{aligned}&5[[f(x),x],x]y[[f(x),x],x]+15[[f(x),x],x][y,x][f(x),x]\\&+4[f(x),x][y,x][[f(x),x],x]+6[f(x),x][[y,x],x][f(x),x]\\&+4[f(x),x][[[y,x],x],x]f(x)+15[[f(x),x],x][[y,x],x]f(x)\\&=0,\quad x,y\in R.\end{aligned}\tag{33}$$

And, combining $4\times(19)$ with (22) it follows that

$$\begin{aligned}&10[[f(x),x],x]y[[f(x),x],x]+8[[f(x),x],x][y,x][f(x),x]\\&+8[f(x),x][[y,x],x][f(x),x]+4f(x)[[[y,x],x],x][f(x),x]\\&+20[f(x),x][y,x][[f(x),x],x]+15f(x)[[y,x],x][[f(x),x],x]\\&=0,\quad x,y\in R.\end{aligned}\tag{34}$$

Subtracting (30) from (34), we have

$$\begin{aligned}&5[[f(x),x],x]y[[f(x),x],x]+4[[f(x),x],x][y,x][f(x),x]\\&+5[f(x),x][y,x][[f(x),x],x]+2[f(x),x][[y,x],x][f(x),x]\\&=0,\quad x,y\in R.\end{aligned}\tag{35}$$

Applying (LA) to (23), we get

$$
\begin{aligned}
&10[[f(x),x],x]y[f(x),x]+20[[f(x),x],x][y,x]f(x)\\
&+5[f(x),x]y[[f(x),x],x]+15[f(x),x][y,x][f(x),x]\\
&+15[f(x),x][[y,x],x]f(x)+4f(x)[y,x][[f(x),x],x]\\
&+6f(x)[[y,x],x][f(x),x]+4f(x)[[[y,x],x],x]f(x)\\
&=0,\quad x,y\in R. \qquad (36)
\end{aligned}
$$

On the other hand, taking $4\times(20)$ from (32) we have

$$
\begin{aligned}
&10[[f(x),x],x]y[[f(x),x],x]+20[[f(x),x],x][y,x][f(x),x]\\
&+8[f(x),x][y,x][[f(x),x],x]+8[f(x),x][[y,x],x][f(x),x]\\
&+15[[f(x),x],x][[y,x],x]f(x)+4[f(x),x][[[y,x],x],x]f(x)\\
&=0,\quad x,y\in R. \qquad (37)
\end{aligned}
$$

Subtracting (33) from (37), we obtain

$$
\begin{aligned}
&5[[f(x),x],x]y[[f(x),x],x]+5[[f(x),x],x][y,x][f(x),x]\\
&+4[f(x),x][y,x][[f(x),x],x]+2[f(x),x][[y,x],x][f(x),x]\\
&=0,\quad x,y\in R. \qquad (38)
\end{aligned}
$$

And again, comparing (35) with (38), it follows that

$$
[[f(x),x],x][y,x][f(x),x]-[f(x),x][y,x][[f(x),x],x]=0,\quad x,y\in R. \qquad (39)
$$

And also, applying (RA) to (25), we get

$$
\begin{aligned}
&10[f(x),x]y[[f(x),x],x]+20f(x)[y,x][[f(x),x],x]\\
&+5[[f(x),x],x]y[f(x),x]+15[f(x),x][y,x][f(x),x]\\
&+15f(x)[[y,x],x][f(x),x]+4[[f(x),x],x][y,x]f(x)\\
&+6[f(x),x][[y,x],x]f(x)+4f(x)[[[y,x],x],x]f(x)\\
&=0,\quad x,y\in R. \qquad (40)
\end{aligned}
$$

Thus, taking (40) from (36) we have

$$\begin{aligned}&5[[f(x),x],x]y[f(x),x]+16[[f(x),x],x][y,x]f(x)\\&-5[f(x),x]y[[f(x),x],x]+9[f(x),x][[y,x],x]f(x)\\&-16f(x)[y,x][[f(x),x],x]-9f(x)[[y,x],x][f(x),x]\\&=0,\quad x,y\in R.\end{aligned}\tag{41}$$

Applying (RA) to (41), we get

$$\begin{aligned}&5[[f(x),x],x]y[[f(x),x],x]+16[[f(x),x],x][y,x][f(x),x]\\&+9[f(x),x][[y,x],x][f(x),x]-9f(x)[[y,x],x][[f(x),x],x]\\&=0,\quad x,y\in R.\end{aligned}\tag{42}$$

Hence, comparing (22) with (32), one obtains

$$\begin{aligned}&15([[f(x),x],x][[y,x],x]f(x)-f(x)[[y,x],x][[f(x),x],x])\\&+4([[f(x),x],x][[[y,x],x],x]e(x)-e(x)[[[y,x],x],x][[f(x),x],x])\\&=0,\quad x,y\in R.\end{aligned}\tag{43}$$

And also, subtracting (16)×$[f(x),x]$ from $[f(x),x]$×(17) we arrive at

$$\begin{aligned}&5([f(x),x][[[y,x],x],x]f(x)-f(x)[[[y,x],x],x][f(x),x])\\&+([f(x),x][[[[y,x],x],x],x]e(x)-e(x)[[[[y,x],x],x],x][f(x),x])\\&=0,\quad x,y\in R.\end{aligned}\tag{44}$$

(20) added to (19) gives

$$\begin{aligned}&([[f(x),x],x][[[y,x],x],x]e(x)-e(x)[[[y,x],x],x][[f(x),x],x])\\&-([f(x),x][[[y,x],x],x]f(x)-f(x)[[[y,x],x],x][f(x),x])\\&=0,\quad x,y\in R.\end{aligned}\tag{45}$$

On the other hand, applying (LA) to (41) we have

$$\begin{aligned}&-5[[f(x),x],x]y[[f(x),x],x]-16[f(x),x][y,x][[f(x),x],x]\\&-9[f(x),x][[y,x],x][f(x),x]+9[[f(x),x],x][[y,x],x]f(x)\\&=0,\quad x,y\in R.\end{aligned}\tag{46}$$

And so, combining (42) with (46), and using (39) and the condition $\operatorname{char}(R)\neq 3$ one obtains

$$[[f(x),x],x][[y,x],x]f(x)-f(x)[[y,x],x][[f(x),x],x]=0,\quad x,y\in R.\tag{47}$$

Comparing (43) with (47), and using the assumption that $\operatorname{char}(R)\neq 2$ we arrive at

$$\begin{aligned}&[[f(x),x],x][[[y,x],x],x]e(x)-e(x)[[[y,x],x],x][[f(x),x],x]\\&=0,\quad x,y\in R.\end{aligned}\tag{48}$$

And also, it follows from (45), (48) that

$$[f(x),x][[[y,x],x],x]f(x)-f(x)[[[y,x],x],x][f(x),x]=0,\quad x,y\in R.\tag{49}$$

From now on, we use the relations (47), (48), and (49) without specific reference.

From (35) and (39), we have

$$\begin{aligned}&5[[f(x),x],x]y[[f(x),x],x]+9[[f(x),x],x][y,x][f(x),x]\\&+2[f(x),x][[y,x],x][f(x),x]=0,\quad x,y\in R.\end{aligned}\tag{50}$$

And, subtracting $2\times$ (42) from $9\times$(50) it follows that

$$\begin{aligned}&35[[f(x),x],x]y[[f(x),x],x]+49[[f(x),x],x][y,x][f(x),x]\\&+18[[f(x),x],x][[y,x],x]f(x)=0,\quad x,y\in R.\end{aligned}\tag{51}$$

And, replacing $[y,x]$ for y in (36), it is obvious that

$$15[[f(x),x],x][y,x][f(x),x]+24[[f(x),x],x][[y,x],x]f(x)$$
$$+15[f(x),x][[y,x],x][f(x),x]+21f(x)[[[y,x],x],x][f(x),x]$$
$$+4f(x)[[[[y,x],x],x],x]f(x)=0, \quad x,y\in R. \tag{52}$$

Applying (LA) to (16), we get

$$10[[f(x),x],x][[y,x],x]+5[f(x),x][[[y,x],x],x]$$
$$+f(x)[[[[y,x],x],x],x]=0, \quad x,y\in R. \tag{53}$$

Thus, multiplying (53) by $f(x)$ on the right we have

$$10[[f(x),x],x][[y,x],x]f(x)+5[f(x),x][[[y,x],x],x]f(x)$$
$$+f(x)[[[[y,x],x],x],x]f(x)=0, \quad x,y\in R. \tag{54}$$

And so, taking $4\times$ (54) from (52), we obtain

$$15[[f(x),x],x][y,x][f(x),x]-16[[f(x),x],x][[y,x],x]f(x)$$
$$+15[f(x),x][[y,x],x][f(x),x]+f(x)[[[y,x],x],x][f(x),x]$$
$$=0, \quad x,y\in R. \tag{55}$$

On the other hand, subtracting $4\times$(55) from (33) it follows that

$$5[[f(x),x],x]y[[f(x),x],x]-41[[f(x),x],x][y,x][f(x),x]$$
$$+79[[f(x),x],x][[y,x],x]f(x)-54[f(x),x][[y,x],x][f(x),x]$$
$$=0, \quad x,y\in R. \tag{56}$$

Combining $27\times$(50) with (56), we arrive at

$$140[[f(x),x],x]y[[f(x),x],x]+202[[f(x),x],x][y,x][f(x),x]$$
$$+79[[f(x),x],x][[y,x],x]f(x)=0, \quad x,y\in R. \tag{57}$$

Furthemore, taking 18×(57) from 79×(51), and using the condition char$(R) \neq 5$ it follows that

$$49[[f(x),x],x]y[[f(x),x],x]+47[[f(x),x],x][y,x][f(x),x] \\ =0, \quad x,y \in R. \tag{58}$$

Replacing y by $[y,x]$ in (16)×$f(x)$, we have

$$10[[f(x),x],x][[y,x],x]f(x)+10[f(x),x][[[y,x],x],x]f(x) \\ +5f(x)[[[[y,x],x],x],x]f(x)+e(x)[[[[[y,x],x],x],x],x]f(x) \\ =0, \quad x,y \in R. \tag{59}$$

But, applying (RA) to (17), we get

$$10[[y,x],x][[f(x),x],x]+5[[[y,x],x],x][f(x),x]+[[[[y,x],x],x],x]f(x) \\ =0, \quad x,y \in R. \tag{60}$$

And also, putting $[y,x]$ instead of y in (60) we arrive at

$$10[[[y,x],x],x][[f(x),x],x]+5[[[[y,x],x],x],x][f(x),x] \\ +[[[[[y,x],x],x],x],x]f(x)=0, \quad x,y \in R. \tag{61}$$

Comparing (59) with $e(x)$×(61), we obtain

$$10[[f(x),x],x][[y,x],x]f(x)+10[f(x),x][[[y,x],x],x]f(x) \\ +5f(x)[[[[y,x],x],x],x]f(x)-10e(x)[[[y,x],x],x][[f(x),x],x] \\ -5e(x)[[[[y,x],x],x],x][f(x),x]=0, \quad x,y \in R. \tag{62}$$

And, combining (16)×$5[f(x),x]$ with (62) it follows that

$$50[[f(x),x],x][y,x][f(x),x]+10[[f(x),x],x][[y,x],x]f(x) \\ -10[[f(x),x],x][[[y,x],x],x]e(x)+50[f(x),x][[y,x],x][f(x),x] \\ +35[f(x),x][[[y,x],x],x]f(x)+5f(x)[[[[y,x],x],x],x]f(x) \\ =0, \quad x,y \in R. \tag{63}$$

Taking 5×(54) from (63), we have

$$\begin{aligned}&50[[f(x),x],x][y,x][f(x),x]-40[[f(x),x],x][[y,x],x]f(x)\\&-10[[f(x),x],x][[[y,x],x],x]e(x)+50[f(x),x][[y,x],x][f(x),x]\\&+10[f(x),x][[[y,x],x],x]f(x)=0,\quad x,y\in R. \qquad (64)\end{aligned}$$

And again, subtracting (64) from 10×(55), and using char(R) $\neq 2,5$ we get

$$\begin{aligned}&10[[f(x),x],x][y,x][f(x),x]-12[[f(x),x],x][[y,x],x]f(x)\\&+[[f(x),x],x][[[y,x],x],x]e(x)+10[f(x),x][[y,x],x][f(x),x]\\&=0,\quad x,y\in R. \qquad (65)\end{aligned}$$

Comparing 5× (50) with (65), we obtain

$$\begin{aligned}&25[[f(x),x],x]y[[f(x),x],x]+35[[f(x),x],x][y,x][f(x),x]\\&+12[[f(x),x],x][[y,x],x]f(x)-[[f(x),x],x][[[y,x],x],x]e(x)\\&=0,\quad x,y\in R. \qquad (66)\end{aligned}$$

And again, (22) added to 4×(66) makes the following:

$$\begin{aligned}&110[[f(x),x],x]y[[f(x),x],x]+160[[f(x),x],x][y,x][f(x),x]\\&+63[[f(x),x],x][[y,x],x]f(x)=0,\quad x,y\in R. \qquad (67)\end{aligned}$$

Thus, taking 2×(67) from 7×(51), we have

$$\begin{aligned}&25[[f(x),x],x]y[[f(x),x],x]+23[[f(x),x],x][y,x][f(x),x]\\&=0,\quad x,y\in R. \qquad (68)\end{aligned}$$

Hence, subtracting $23\times(58)$ from $47\times(68)$, and using the assumption that $\text{char}(R) \neq 2, 3$ we arrive at

$$[[f(x), x], x]y[[f(x), x], x] = 0, \quad x, y \in R. \tag{69}$$

Therefore, by semiprimeness of R it is immediate from (69) that

$$[[f(x), x], x] = 0, \quad x, y \in R. \tag{70}$$

Therefore by Theorem 2 in [2] we obtain $D = 0$. The proof of the theorem is complete.

Let $\text{Inv}(R)$ denote the set of all invertible elements in a ring R with identity.

THEOREM 2.2. *Let R be a noncommutative semiprime ring with $char(R) \neq 2$, 3, and 5. Suppose there exists a derivation $D : R \longrightarrow R$ such that the mapping $x \longmapsto [[[Dx, x], x], x]$ is centralizing on R. And assume for some $m \in N$, $(Dx)^m = 0$ for all $x \in R$. Then we have $D = 0$.*

PROOF. The given assumptions and the relation (70) satisfy the conditions of Theorem 2.1 in [1]. And so, we have $D = 0$.

Let $[R, R]$ denote the set of all commutators $[x, y]$ for $x, y \in R$.

THEOREM 2.3. *Let R be a noncommutative semiprime ring with identity and $char(R) \neq 2$, 3, and 5. Suppose there exists a derivation $D : R \longrightarrow R$ such that the mapping $x \longmapsto [[[Dx, x], x], x]$ is centralizing on R. If $Inv(R) \cap [R, R]$ is nonempty, then we have $D = 0$.*

PROOF. The given assumptions and the relation (70) satisfy the conditions of Theorem 2.2 in [1]. And so, we have $D = 0$.

The following theorem is due to Vukman [3].

THEOREM 2.4. *Let A be a noncommutative Banach algebra, and let $D : A \longrightarrow A$ be a continuous linear Jordan derivation. If $[[[[Dx, x], x], x], x] \in rad(A)$ for all $x \in A$, then D maps A into rad(A).*

PROOF. The proof goes through in the same way as the proof of Theorem 2 in Vukman's paper [3].

When a Banach algebra is semisimple, one can prove the following result.

THEOREM 2.5. *Let A be a noncommutative semisimple Banach algebra. Suppose there exists a linear Jordan derivation $D : A \longrightarrow A$, such that the mapping $x \mapsto [[[Dx, x], x], x]$ is commuting on A. In this case $D = 0$.*

REFERENCES

1. K.W. Jun and B.D. Kim, *On derivations in semiprime rings and Banach algebras*, Preprint (1994).
2. J. Vukman, *Two results concerning symmetric bi-derivations on prime rings*, Aequationes Math. **40** (1990), 181-189.
3. J. Vukman, *On derivations in prime rings and Banach algebras*, Proc. Amer. Math. Soc. **116** (1992), 877-884.

*

DEPARTMENT OF MATHEMATICS
KANGNUNG NATIONAL UNIVERSITY
KANGNUNG, 210-702, KOREA

**

DEPARTMENT OF MATHEMATICS EDUCATION
KONGJU NATIONAL TEACHER'S COLLEGE
KONGJU, 314-060, KOREA