JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume 7, June 1994

Some Properties of Generalized Fractions

DONG-SOO LEE* AND SANG-CHO CHUNG

ABSTRACT. Let A be a commutative ring with identity and M an A-module. When U_n is a triangular subset of A^n, Sharp and Zakeri defined a module of generalized fractions $U_n^{-n}M$. In [SZ3], they described a relation of the Monomial Conjecture and a module of generalized fractions under the condition of a Noetherian local ring. In this paper, we investigate some properties of non-zero generalized fractions and give a generalization of results of Sharp and Zakeri for an arbitrary ring.

0. Introduction

Let A be a commutative ring with identity and M an A-module. Put $U_{d+1} = \{(b_1, \ldots, b_d, 1) \in A^{d+1} : b_1, \ldots, b_d$ forms a system of parameters for $A\}$, where A is a Noetherian local ring of dimension d. Then $U_{d+1}^{-d-1}A$ becomes a module of generalized fractions (cf. [SZ1]). Let $\{a_1, \ldots, a_n\}$ be a fixed system of parameters for A. In [SZ3], Sharp and Zakeri described a relation of the Monomial Conjecture and a module of generalized fractions. They proved that $a_1^{t-1} \cdots a_d^{t-1} \notin (a_1^t, \ldots, a_d^t)A$ for all $t \geq 1$ if and only if $\dfrac{1}{(a_1, \ldots, a_d, 1)} \neq 0$ in $U_{d+1}^{-d-1}A$.

So we take an interest in properties of non-zero generalized fractions. The purpose of this paper is to investigate some properties of non-zero generalized fractions and to give a generalization of results of Sharp and Zakeri (Theorem 3.2). That is, fix a sequence $a_1, \ldots, a_n$ of

*The first author was partially supported by BSRI-94-1427, Ministry of Education.

Received by the editors on June 30, 1994.

1980 *Mathematics subject classifications*: Primary 13C05.

elements of an arbitrary ring A. Then we have that $ma_1^{t-1} \cdots a_n^{t-1} \notin (a_1^t, \dots, a_{n-1}^t)M$ for all $t \geq 1$ if and only if $\frac{m}{(a_1, \dots, a_n)} \neq 0$ in $U(a)_n^{-n}M$, where $U(a)_n = \{(a_1^{\alpha_1}, \dots, a_n^{\alpha_n}) \in A^n : \alpha_1, \dots, \alpha_n$ are positive integers $\}$.

1. Preliminaries

Throughout this paper, A is a commutative ring with identity. M denotes an A -module. We use ^{T} to denote matrix transpose, n to denote a positive integer, and $D_n(A)$ to denote the set of $n \times n$ lower triangular matrices over A. For H $\in D_n(A)$, |H| denotes the determinant of H. N denotes the set of positive integers. Let $(a_1, \dots, a_i)A$ be the ideal of A which is generated by $\{a_1, \dots, a_i\}$ and let $(a_1, \dots, a_i)M$ be the submodule of M which is generated by $\{a_j m : j = 1, \dots, i \text{ and } m \in M\}$.

DEFINITION 1.1 [SZ1]. A *triangular subset of* A^n is a non-empty subset U_n of A^n such that

(i) if $(a_1, \dots, a_n) \in U_n$, then $(a_1^{\alpha_1}, \dots, a_n^{\alpha_n}) \in U_n$ for all choices of positive integers $\alpha_1, \dots, \alpha_n$, and
(ii) if $(a_1, \dots, a_n) \in U_n$ and $(b_1, \dots, b_n) \in U_n$, then there exist $(c_1, \dots, c_n) \in U_n$ and H, K $\in D_n(A)$ such that H$[a_1 \ \dots \ a_n]^T = [c_1 \ \dots \ c_n]^T =$ K$[b_1 \ \dots \ b_n]^T$.

Let U_n be a triangular subset of A^n. The *module of generalized fractions* $U_n^{-n}M$ *of* M *with respect to* U_n is a module, whose elements, called *generalized fractions* , have the form

$$\frac{m}{(a_1, \dots, a_n)},$$

where $m \in M$ and $(a_1, \dots, a_n) \in U_n$, satisfying the following condition:

Let $m,\ m' \in M$ and $(a_1,\ldots,a_n),\ (b_1,\ldots,b_n) \in U_n$. Then $\dfrac{m}{(a_1,\ldots,a_n)} = \dfrac{m'}{(b_1,\ldots,b_n)}$ if and only if there exist $(c_1,\ldots,c_n) \in U_n$ and $\mathrm{H},\ \mathrm{K} \in D_n(A)$ such that

$$\mathrm{H}[a_1\ \ldots\ a_n]^T = [c_1\ \ldots\ c_n]^T = \mathrm{K}[b_1\ \ldots\ b_n]^T \text{ and}$$
$$|\mathrm{H}|m - |\mathrm{K}|m' \in (c_1,\ldots,c_{n-1})M.$$

The reader is referred to [SZ1, SZ2] for more details of the construction.

Let $\overline{U}_n = \{(a_1,\ldots,a_i,1,\ldots,1) \in A^n$: for all i $(0 \le i \le n)$, there exist $a_{i+1},\ldots,a_n \in A$ such that $(a_1,\ldots,a_i,a_{i+1},\ldots,a_n) \in U_n\}$. This is a triangular subset of A^n and is called the *expansion* of U_n. Then, by [SZ1, 3.2], we may assume without loss of the generality that U_n is *expanded*, *i.e.*, $U_n = \overline{U}_n$, when we consider a module of generalized fractions for M with respect to U_n.

A given triangular subset U_n of A^n, let U_t $(1 \le t < n)$ be a *restriction* of U_n to A^t, *i.e.*, $U_t = \{(a_1,\ldots,a_t) \in A^t$: there exist $a_{t+1},\ldots,a_n \in A$ such that $(a_1,\ldots,a_t,a_{t+1},\ldots,a_n) \in U_n\}$. Then clearly U_t is a triangular subset of A^t.

We say that a sequence of elements $a_1,\ldots,a_n$ of A is a *poor M-sequence* if a_i is not a zerodivisor on $M/(a_1,\ldots,a_{i-1})M$ for each $i = 1,\ldots,n$; it is an *M-sequence* if, in addition, $M \neq (a_1,\ldots,a_n)M$.

LEMMA 1.2 [HS, SZ1, SZ2, O]. *Suppose that M is an A-module. Let U_n be expanded. Let $(a_1,\ldots,a_n)$ and $(b_1,\ldots,b_n)$ be elements of U_n such that* $\mathrm{H}[a_1\ \ldots\ a_n]^T = [b_1\ \ldots\ b_n]^T$ *for some* $\mathrm{H} \in D_n(A)$. *Then we have the following.*

(1) $\dfrac{m}{(a_1,\ldots,a_n)} = \dfrac{|\mathrm{H}|m}{(b_1,\ldots,b_n)} = \dfrac{a_1^{\alpha_1-1}\cdots a_n^{\alpha_n-1}m}{(a_1^{\alpha_1},\ldots,a_n^{\alpha_n})}$ *for any* $(\alpha_1,\ldots,\alpha_n) \in N^n$ *and*
$\dfrac{a_n m}{(a_1,\ldots,a_n)} = \dfrac{m}{(a_1,\ldots,a_{n-1},1)}$ *in* $U_n^{-n}M$.

(2) *If* $m \in (a_1, \dots, a_{n-1})M$, *then* $\dfrac{m}{(a_1, \dots, a_n)} = 0$ *in* $U_n^{-n}M$.
In particular, if each member of U_n *is a poor* M*-sequence, then the converse is true.*

(3) *If* $\dfrac{a_n m}{(a_1, \dots, a_n)} = 0$, *then* $\dfrac{m}{(a_1, \dots, a_n)} = 0$ *in* $U_n^{-n}M$.

(4) $\operatorname{Ann}_A\left(\dfrac{m}{(a_1, \dots, a_n)}\right) = \operatorname{Ann}_A\left(\dfrac{m}{(a_1, \dots, a_{n-1}, 1)}\right)$.

(5) *If* $\dfrac{m}{(a_1, \dots, a_n)} \neq 0$ *in* $U_n^{-n}M$, *then* $\dfrac{m}{(a_1, \dots, a_i)} \neq 0$ *in* $U_i^{-i}M$ *for all* i *with* $0 < i < n$.

2. Some properties of non-zero generalized fractions

LEMMA 2.1. *Let M be an A-module. Let* $\dfrac{m}{(a_1, \dots, a_{n+1})} \in U_{n+1}^{-n-1}M$ *and let* α_i, $\beta_i \in N$ *such that* $\alpha_i \geq \beta_i$ *for all* i $(1 \leq i \leq n)$. *Then, for all* $(a_1, \dots, a_n, b_{n+1})$, $(a_1, \dots, a_n, c_{n+1}) \in U_{n+1}$ we have

$$\operatorname{Ann}_A\left(\frac{m}{(a_1^{\alpha_1}, \dots, a_n^{\alpha_n}, b_{n+1})}\right) \subset \operatorname{Ann}_A\left(\frac{m}{(a_1^{\beta_1}, \dots, a_n^{\beta_n}, c_{n+1})}\right).$$

PROOF. From Lemma 1.2(4), we have

$$\begin{aligned} r \in \operatorname{Ann}_A\left(\frac{m}{(a_1^{\alpha_1}, \dots, a_n^{\alpha_n}, b_{n+1})}\right) &= \operatorname{Ann}_A\left(\frac{m}{(a_1^{\alpha_1}, \dots, a_n^{\alpha_n}, 1)}\right) \\ &= \operatorname{Ann}_A\left(\frac{m}{(a_1^{\alpha_1}, \dots, a_n^{\alpha_n}, c_{n+1})}\right). \end{aligned}$$

Hence we get

$$\frac{rm}{(a_1^{\alpha_1}, \dots, a_n^{\alpha_n}, c_{n+1})} = 0.$$

Therefore we have

$$\frac{rma_1^{\alpha_1-\beta_1} \cdots a_n^{\alpha_n-\beta_n}}{(a_1^{\alpha_1}, \dots, a_n^{\alpha_n}, c_{n+1})} = \frac{rm}{(a_1^{\beta_1}, \dots, a_n^{\beta_n}, c_{n+1})} = 0.$$

The next proposition easily follows from Lemma 2.1.

PROPOSITION 2.2. *If* $\frac{m}{(a_1, \dots, a_{n+1})} \neq 0$, *then we have, for all* $\gamma_i \in N$ $(1 \leq i \leq n)$ *and* $(a_1, \dots, a_n, b_{n+1}) \in U_{n+1}$,

$$\frac{m}{(a_1^{\gamma_1}, \dots, a_n^{\gamma_n}, b_{n+1})} \neq 0.$$

COROLLARY 2.3. *Assume that* $\frac{m}{(a_1, \dots, a_{n+1})} \neq 0$ *in* $U_{n+1}^{-n-1}M$ *and* $(a_1, \dots, a_n, b_{n+1})$, $(a_1, \dots, a_n, c_{n+1}) \in U_{n+1}$. *Let* α_i, $\beta_i \in N, i = 1, \dots, n$. *Then*

(1) $\operatorname{Ann}\left(\frac{m}{(a_1^{\alpha_1}, \dots, a_n^{\alpha_n}, b_{n+1})}\right) = \operatorname{Ann}\left(\frac{m}{(a_1^{\beta_1}, \dots, a_n^{\beta_n}, c_{n+1})}\right)$ *if and only if* $\alpha_i = \beta_i$ *for all* $i = 1, \dots, n$.

(2) *If* $\frac{m}{(a_1^{\alpha_1}, \dots, a_n^{\alpha_n}, b_{n+1})} = \frac{m}{(a_1^{\beta_1}, \dots, a_n^{\beta_n}, c_{n+1})}$, *then* $\alpha_i = \beta_i$ *for all* $i = 1, \dots, n$.

In particular, if $b_{n+1} = c_{n+1}$, *then the converse holds.*

PROOF. (1) ($\Leftarrow$) This follows from Lemma 1.2(4).

($\Rightarrow$) Assume that there exists i $(1 \leq i \leq n)$ such that $\alpha_i < \beta_i$, say $\alpha_1 < \beta_1$. Then we get

$$a_1^{\alpha_1} \in \operatorname{Ann}_A\left(\frac{m}{(a_1^{\alpha_1}, \dots, a_n^{\alpha_n}, b_{n+1})}\right) = \operatorname{Ann}_A\left(\frac{m}{(a_1^{\beta_1}, \dots, a_n^{\beta_n}, c_{n+1})}\right).$$

Hence we have

$$\frac{a_1^{\alpha_1}m}{(a_1^{\beta_1}, \dots, a_n^{\beta_n}, c_{n+1})} = \frac{m}{(a_1^{\beta_1-\alpha_1}, a_2^{\beta_2}, \dots, a_n^{\beta_n}, c_{n+1})} = 0.$$

This contradicts to the above Proposition 2.2.

(2) This easily follows from (1).

EXAMPLE 2.4. In Corollary 2.3(2), we need the condition $b_{n+1} = c_{n+1}$. Let $A = M = Z$ and $U_2 = \{(2^\alpha, 3^\beta) \in Z^2 : \alpha, \beta \in N \cup \{0\}\}$. Then we have $\frac{1}{(2^2, 1)} \neq \frac{1}{(2^2, 3)}$. In fact, assume that $\frac{1}{(2^2, 1)} = \frac{1}{(2^2, 3)}$ or $\frac{3-1}{(2^2, 3)} = 0$. Hence we get $\frac{1}{(2, 3)} = 0$ but it is not zero in $U_2^{-2}Z$.

COROLLARY 2.5. *Suppose that* $\frac{m}{(a_1,\dots,a_{n+1})} \neq 0$ *in* $U_{n+1}^{-n-1}M$. *Fix an integer* $t \in N$ *with* $0 \leq t \leq n-1$. *Let* U_{t+1} *be a restriction of* U_{n+1} *to* A^{t+1}. *Then we have*
$\frac{m}{(a_1^{\alpha_1},\dots,a_t^{\alpha_t},b_{t+1})} = \frac{m}{(a_1^{\beta_1},\dots,a_t^{\beta_t},b_{t+1})}$ *in* $U_{t+1}^{-t-1}M$ *if and only if* $\alpha_i = \beta_i$ *for all* $i = 1,\dots,t$.

PROOF. This follows from Lemma 1.2(5) and Corollary 2.3(2).

PROPOSITION 2.6. *Let* M *be an* A*-module. Suppose* $a_n \in \mathrm{Rad}(A)$ *and* $\frac{m}{(a_1,\dots,a_n)} \neq 0$ *in* $U_n^{-n}M$. *Then we have*
$\frac{m}{(a_1^{\alpha_1},\dots,a_n^{\alpha_n})} = \frac{m}{(a_1^{\beta_1},\dots,a_n^{\beta_n})}$ *in* $U_n^{-n}M$ *if and only if* $\alpha_i = \beta_i$ *for all* $i = 1,\dots,n$.

PROOF. We prove only the sufficiency. By Corollary 2.3(2), we may assume that $\alpha_1 = \cdots = \alpha_{n-1} = \beta_1 = \cdots = \beta_{n-1} = 1$. Assume that $\alpha_n > \beta_n$.
Suppose that $\frac{m}{(a_1,\dots,a_{n-1},a_n^{\alpha_n})} = \frac{m}{(a_1,\dots,a_{n-1},a_n^{\beta_n})}$. Then, by the property of the module of generalized fractions, there exist $(b_1,\dots,b_n) \in U_n$ and $\mathrm{H}, \mathrm{K} \in D_n(A)$ such that $\mathrm{H}[a_1 \ \dots \ a_{n-1} \ a_n^{\alpha_n}]^T = [b_1 \ \dots \ b_n]^T = \mathrm{K}[a_1 \ \dots \ a_{n-1} \ a_n^{\beta_n}]^T$ and $|\mathrm{H}|m - |\mathrm{K}|m \in (b_1,\dots,b_{n-1})M$. Let D be the diagonal matrix $diag(b_1,\dots,b_n)$. Hence we have

$$|\mathrm{D}||\mathrm{H}|m - |\mathrm{D}||\mathrm{K}|m \in (b_1^2,\dots,b_{n-1}^2)M.$$

On the other hand, we can rewrite as follows;

$$\mathrm{H}\begin{pmatrix} 1 & & & 0 \\ & \ddots & & \\ & & 1 & \\ 0 & & & a_n^{\alpha_n-\beta_n} \end{pmatrix}\begin{pmatrix} a_1 \\ \vdots \\ a_{n-1} \\ a_n^{\beta_n} \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_{n-1} \\ b_n \end{pmatrix} = \mathrm{K}\begin{pmatrix} a_1 \\ \vdots \\ a_{n-1} \\ a_n^{\beta_n} \end{pmatrix}.$$

Thus by [SZ1, 2.3] we have

$$|\mathrm{D}||\mathrm{H}|a_n^{\alpha_n-\beta_n}m - |\mathrm{D}||\mathrm{K}|m \in (b_1^2, \ldots, b_{n-1}^2)M.$$

Therefore we get

$$|\mathrm{D}||\mathrm{H}|m(1 - a_n^{\alpha_n-\beta_n}) \in (b_1^2, \ldots, b_{n-1}^2)M.$$

Since $a_n \in \mathrm{Rad}(A)$, we obtain

$$|\mathrm{D}||\mathrm{H}|m \in (b_1^2, \ldots, b_{n-1}^2)M.$$

Hence we have the following contradiction;

$$\begin{aligned}\frac{m}{(a_1, \ldots, a_{n-1}, a_n^{\alpha_n})} &= \frac{|\mathrm{H}|m}{(b_1, \ldots, b_n)} = \frac{b_1 \cdots b_n|\mathrm{H}|m}{(b_1^2, \ldots, b_n^2)}\\ &= \frac{|\mathrm{D}||\mathrm{H}|m}{(b_1^2, \ldots, b_n^2)} = 0\end{aligned}$$

by Lemma 1.2(1)(2).

EXAMPLE 2.7. In Proposition 2.6, we cannot omit the condition $a_n \in \mathrm{Rad}(A)$. Under the same assumption as in Example 2.4, we have $\frac{1}{(2,1)} = \frac{1}{(2,3)}$ in $U_2^{-2}Z$, since $\frac{1}{(2,1)} - \frac{1}{(2,3)} = \frac{3-1}{(2,3)} = 0$ by Lemma 1.2(2).

PROPOSITION 2.8. *Let M be an A-module. Let $\frac{m}{(a_1, \ldots, a_{n+1})} \in U_{n+1}^{-n-1}M$. Then we have*

$$\mathrm{Ann}_A\left(\frac{m}{(a_1, \ldots, a_n)}\right) \subset \mathrm{Ann}_A\left(\frac{m}{(a_1, \ldots, a_{n+1})}\right).$$

In particular, if $\frac{m}{(a_1, \ldots, a_n)} \neq 0$ in $U_n^{-n}M$ then the above inclusion is not equal.

PROOF. By Lemma 1.2(5) and (3) the proposition is clear.

REMARK 2.9. If $U_i^{-i}M = 0$ for some $i \in N$, then we have $U_{i+1}^{-i-1}M = 0$ by Proposition 2.8, whenever U_i is a restriction of U_{i+1} to A^i.

PROPOSITION 2.10. *Let M be an A-module. Assume that each member of U_{n+1} is a poor M-sequence. Let $0 \neq x = \dfrac{m}{(a_1^{\alpha_1}, \dots, a_n^{\alpha_n}, a_{n+1})} \in U_{n+1}^{-n-1}M$ and $\mathrm{Ann}_A(x) \supset (a_1, \dots, a_n)A$. Then there exists $m' \in M$ such that $m = m' a_1^{\alpha_1 - 1} \cdots a_n^{\alpha_n - 1}$ with $m' \notin (a_1, \dots, a_n)M$. That is, we have*

$$\frac{m}{(a_1^{\alpha_1}, \dots, a_n^{\alpha_n}, a_{n+1})} = \frac{m'}{(a_1, \dots, a_n, a_{n+1})} \text{ where } m' \notin (a_1, \dots, a_n)M.$$

PROOF. Note that $\mathrm{Ann}_A(x) \supset (a_1, \dots, a_n)A$.

Suppose that $m = m' a_1^{\gamma_1} \cdots a_n^{\gamma_n}$ for some $(\gamma_i) \in (N \cup \{0\})^n$ and $m' \in (a_1, \dots, a_n)M$. Then since $m \notin (a_1^{\alpha_1}, \dots, a_n^{\alpha_n})M$ by Lemma 1.2(2), we have $\gamma_i < \alpha_i$ for all i and we can find $b_i \in M$ such that $m = \sum_{i=1}^{l} b_i a_1^{t_{i1}} \cdots a_n^{t_{in}}$ with $b_i \notin (a_1, \dots, a_n)M$. Here we may assume that $t_{ij} < \alpha_j$ for all i, j $(1 \leq i \leq l,\ 1 \leq j \leq n)$ by Lemma 1.2(2). Hence we have

$$\begin{aligned}
&\frac{m}{(a_1^{\alpha_1}, \dots, a_n^{\alpha_n}, a_{n+1})} \\
=&\frac{b_1 a_1^{t_{11}} \cdots a_n^{t_{1n}}}{(a_1^{\alpha_1}, \dots, a_n^{\alpha_n}, a_{n+1})} + \cdots + \frac{b_l a_1^{t_{l1}} \cdots a_n^{t_{ln}}}{(a_1^{\alpha_1}, \dots, a_n^{\alpha_n}, a_{n+1})} \\
=&\frac{b_1}{(a_1^{\alpha_1 - t_{11}}, \dots, a_n^{\alpha_n - t_{1n}}, a_{n+1})} + \cdots + \frac{b_l}{(a_1^{\alpha_1 - t_{l1}}, \dots, a_n^{\alpha_n - t_{ln}}, a_{n+1})}.
\end{aligned}$$

Put $\beta_1 = \max\{\alpha_1 - t_{11}, \dots, \alpha_1 - t_{l1}\}$. Now by multiplication with $a_1^{\beta_1 - 1}$ and Lemma 1.2(2), we have

$$\begin{aligned}
\frac{a_1^{\beta_1 - 1} m}{(a_1^{\alpha_1}, \dots, a_n^{\alpha_n}, a_{n+1})} &= \frac{b_1}{(a_1, a_2^{\alpha_2 - t_{12}}, \dots, a_n^{\alpha_n - t_{1n}}, a_{n+1})} + \cdots \\
&+ \frac{b_s}{(a_1, a_2^{\alpha_2 - t_{s2}}, \dots, a_n^{\alpha_n - t_{sn}}, a_{n+1})}
\end{aligned}$$

by renumbering of non-zero components ($s \leq l$). Continue the same way, until we obtain $a_1^{\beta_1-1}, \ldots, a_n^{\beta_n-1} \in A$ and $b_1 \notin (a_1, \ldots, a_n)M$ such that

$$\frac{a_1^{\beta_1-1} \cdots a_n^{\beta_n-1}m}{(a_1^{\alpha_1}, \ldots, a_n^{\alpha_n}, a_{n+1})} = \frac{b_1}{(a_1, \ldots, a_n, a_{n+1})} \neq 0,$$

by Lemma 1.2(2). If $\beta_j \neq 1$ for some j, then we have the following contradiction

$$a_1^{\beta_1-1} \cdots a_n^{\beta_n-1} \notin \mathrm{Ann}_A\left(\frac{m}{(a_1^{\alpha_1}, \ldots, a_n^{\alpha_n}, a_{n+1})}\right).$$

Hence we must have $\beta_j = 1$ for all $j = 1, \ldots, n$. That is

$$t_{ij} = \alpha_j - 1 \text{ for all } i,\ j\ (1 \leq i \leq l,\ 1 \leq j \leq n).$$

Then by the hypothesis we have

$$b_1 + \cdots + b_l \notin (a_1, \ldots, a_n)M.$$

3. A generalization of results of Sharp and Zakeri

From now on, let $\{a_1, \ldots, a_n\}$ be a fixed sequence of elements of A. Then obviously

$$U(a)_n = \{(a_1^{\alpha_1}, \ldots, a_n^{\alpha_n}) \in A^n : \alpha_i \in N,\ i = 1, \ldots, n\}$$

and

$$U(a)_n[1] = \{(a_1^{\alpha_1}, \ldots, a_n^{\alpha_n}, 1) \in A^{n+1} : \alpha_i \in N,\ i = 1, \ldots, n\}$$

are triangular subsets of A^n and A^{n+1}.

Let $(\alpha), (\beta) \in N^n$ with $(\alpha) = (\alpha_1, \ldots, \alpha_n)$. Then $(\alpha) \geq$ (or $>$)(β) is defined as $\alpha_i \geq$ (or $>$)β_i for all i ($1 \leq i \leq n$).

LEMMA 3.1 [O, 1.7 and SH, 2.8]. *Let A be a ring. Let M be an A-module. Then, in $U(a)_n^{-n}M$*

$$\frac{m}{(a_1^{\alpha_1},\ldots,a_n^{\alpha_n})}=\frac{m'}{(a_1^{\beta_1},\ldots,a_n^{\beta_n})}$$

$$\Updownarrow$$

there exists $\gamma \geq \alpha_i, \beta_i$ $(1 \leq i \leq n)$ such that

$$a_1^{\gamma-\alpha_1}\cdots a_n^{\gamma-\alpha_n}m - a_1^{\gamma-\beta_1}\cdots a_n^{\gamma-\beta_n}m' \in (a_1^{\gamma},\ldots,a_{n-1}^{\gamma})M.$$

THEOREM 3.2. *Let A be a ring and M an A-module. Let $(\alpha),(\beta) \in (N \cup \{0\})^{n-1}$ and $\alpha_n \in N \cup \{0\}$. Then the following conditions are equivalent.*

(1) $\dfrac{m}{(a_1,\ldots,a_n)} \neq 0$ *in* $U(a)_n^{-n}M$.

(2) *For all $a_n^{\alpha_n}$, (α) and (β) such that $(\beta) > (\alpha)$,*

$$ma_1^{\alpha_1}\cdots a_n^{\alpha_n} \notin (a_1^{\beta_1},\ldots,a_{n-1}^{\beta_{n-1}})M.$$

(3) *For all $t \geq 1$, $ma_1^{t-1}\cdots a_n^{t-1} \notin (a_1^{t},\ldots,a_{n-1}^{t})M$.*

PROOF. (1)⇒(2) Suppose that, for some $(\beta) > (\alpha)$, $ma_1^{\alpha_1}\cdots a_n^{\alpha_n} \in (a_1^{\beta_1},\ldots,a_{n-1}^{\beta_{n-1}})M$. Then we have

$$\begin{aligned}\frac{m}{(a_1,\ldots,a_n)} &= \frac{ma_1^{\beta_1-1}\cdots a_{n-1}^{\beta_{n-1}-1}a_n^{\alpha_n}}{(a_1^{\beta_1},\ldots,a_{n-1}^{\beta_{n-1}},a_n^{\alpha_n+1})}\\ &= \frac{ma_1^{\alpha_1}\cdots a_n^{\alpha_n}a_1^{\beta_1-\alpha_1-1}\cdots a_{n-1}^{\beta_{n-1}-\alpha_{n-1}-1}}{(a_1^{\beta_1},\ldots,a_{n-1}^{\beta_{n-1}},a_n^{\alpha_n+1})}=0\end{aligned}$$

by Lemma 1.2(2).

(3)⇒(1) Let $\dfrac{m}{(a_1,\ldots,a_n)} = 0$ in $U(a)_n^{-n}M$. Then, from Lemma 3.1, there exists $\gamma \in N$ such that $ma_1^{\gamma-1}\cdots a_n^{\gamma-1} \in (a_1^{\gamma},\ldots,a_{n-1}^{\gamma})M$. This contradicts to the hypothesis.

COROLLARY 3.3 [Ho, Remark 5 and SZ3, 4.1]. *Let A be a Noetherian local ring of dimension d. Put $U_{d+1} = \{(b_1, \ldots, b_d, 1) \in A^{d+1} : b_1, \ldots, b_d$ forms a system of parameters for $A\}$. Let $\{a_1, \ldots, a_d\}$ be a fixed system of parameters for A. Then the following conditions are equivalent.*

(1) $a_1^{t-1} \cdots a_d^{t-1} \notin (a_1^t, \ldots, a_d^t)A$ *for all* $t \geq 1$, *i.e., the Monomial Conjecture holds.*

(2) *Let* $(\alpha), (\beta_1), \ldots, (\beta_n) \in N^d$. *Then we have* $a^{(\alpha)} \in (a^{(\beta_1)}, \ldots, a^{(\beta_n)})A$ *if and only if there exists* i $(1 \leq i \leq n)$ *such that* $(\alpha) \geq (\beta_i)$, *where* $a^{(\alpha)} = a_1^{\alpha_1} \cdots a_d^{\alpha_d}$ *with* $(\alpha) = (\alpha_1, \ldots, \alpha_d)$.

(3) $\dfrac{1}{(a_1, \ldots, a_d, 1)} \neq 0$ *in* $U_{d+1}^{-d-1}A$.

PROOF. In the Theorem 3.2, let $n-1 = d$ and $a_n = 1$. Put $M = A$ and $m = 1$. Then Corollary 3.3 is a special case of Theorem 3.2, since $U(a)_{d+1}^{-d-1}A \cong U(a)_d[1]^{-d-1}A \cong U_{d+1}^{-d-1}A$ by [SZ3, 3.6].

REFERENCES

[HS] M. A. Hamieh and R. Y. Sharp, *Krull dimension and generalized fractions*, Proc. Edinburgh Math. Soc., **28** (1985), 349–353.

[Ho] M. Hochster, *Contracted ideals from integral extensions of regular rings*, Nagoya Math. J., **51** (1973), 25–43.

[O] L. O'carroll, *On the generalized fractions of Sharp and Zakeri*, J. London Math. Soc., (2) **28** (1983), 417–427.

[SH] R. Y. Sharp and M. A. Hamieh, *Lengths of certain generalized fractions*, J. Pure Appl. Algebra, **38** (1985), 323–336.

[SZ1] R. Y. Sharp and H. Zakeri, *Modules of generalized fractions*, Mathematika, **29** (1982), 32–41.

[SZ2] R. Y. Sharp and H. Zakeri, *Modules of generalized fractions and balanced big Cohen-Macaulay modules*, Commutative Algebra: Durham 1981, London Math. Soc. Lecture Note Ser., 72 (Cambridge University Press, 1982), 61–82.

[SZ3] R. Y. Sharp and H. Zakeri, *Local cohomology and modules of generalized fractions*, Mathematika, **29** (1982), 296–306.

DEPARTMENT OF MATHEMATICS,
CHUNGNAM NATIONAL UNIVERSITY,
TAEJON 305-764, KOREA