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A Boundary Element Method for Nonlinear Boundary Value Problems

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ABSTRACT. We consider a numerical scheme for solving a nonlinear boundary integral equation (BIE) obtained by reformulation the nonlinear boundary value problem (BVP). We give a simple alternative to the standard collocation method for the nonlinear BIE. This method consists of one conventional linear system and another coupled linear system resulting from an auxiliary BIE which is obtained by differentiating both side of the nonlinear interior integral equations. We obtain an analogue BIE through the perturbation of the fundamental solution of Laplace's equation. We procure the superconvergence of approximate solutions.

1. Introduction

The major advantage of the boundary element method (BEM) is that the number of the resulting simultaneous equations depends only upon the discretization of the boundary of the domain and that some approximation technique can be employed to represent the solution over the boundary elements. Thus the problem can be treated with one less dimension. The numerical analysis of this method in two dimension is quite completely studied in [1] for the collocation methods and in [9] for Galerkin methods. Further, there exist a lot of advanced programs available to computations.

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Recently, the analysis of the collocation BEM was extended to nonlinear boundary value problems (BVP) [9], where the governing equation itself is linear but the boundary conditions are nonlinear. This kind of nonlinear problems occurs frequently in steady-state heat transfer, where the boundary has a variable thermal heat conductivity or the body obeys the Newton's law of cooling [3]. Also some electromagnetic problems contain nonlinearities in the boundary conditions [4, 10]. It is noted that some nonlinear differential equations can be transformed in this form by the Kirchhoff transformation (See, for example [3] or [10]).

In this paper, we study the nonlinear boundary value problem :

$$\Delta u(x) = 0, \qquad x \in \Omega, \tag{1.1}$$

$$\frac{\partial u(x)}{\partial n_x} = -\, g(x, u(x)) + f(x), \qquad x \in \Gamma, \tag{1.2}$$

where Γ is a smooth simple closed curve, Ω is a two–dimensional region enclosed by Γ such that the transfinite diameter [7] of Γ differs from one, and n_x is the outer normal to Γ at x. The given functions $f(x)$ and $g(x, u(x))$ will be specified later.

We first reformulate the nonlinear BVP (1.1)–(1.2) as a nonlinear boundary integral equation(BIE) to find the numerical solution. The Greens representation formula for harmonic functions together with the jump relations in two dimension gives the following relation :

$$\fint_\Gamma u(y)\Lambda(x-y)\, ds_y - \int_\Gamma \frac{\partial u(y)}{\partial n_y}\lambda(x-y)\, ds_y = \begin{cases} u(x), & x \in \Omega, \\ \frac{1}{2}u(x), & x \in \Gamma. \end{cases} \tag{1.3}$$

where

$$\lambda(x-y) = \frac{1}{2\pi}\log|x-y| \text{ and } \Lambda(x-y) = \frac{\partial \lambda(x-y)}{\partial n_y}$$

and the notation $⨍_\Gamma$ stands for the Cauchy's principal value integral. Substituting the boundary condition (1.2) in the second equation of (1.3), we have a nonlinear mixed Hammerstein type BIE :

$$\begin{aligned}\frac{1}{2}u(x) &= ⨍_\Gamma u(y)\Lambda(x-y)\,ds_y \\ &\quad - \int_\Gamma (f(y) - g(y,u(y))\lambda(x-y))\,ds_y, \quad x \in \Gamma. \end{aligned} \tag{1.4}$$

Numerical analysis of the equation (1.4) was recently initiated in [9] using the Galerkin method and the numerical solvability of (1.4) was studied in [8] for Nyström and collocation methods, respectively. Most recently error estimates obtained for the Galerkin and collocation methods in [8, 9] are improved and generalized in [5]. The superconvergence of approximate solutions is obtained by the product collocation method (PCM) [6].

The first equation of (1.3) provides a continuous representation of $u(x)$ as a function of the boundary value of Λ and λ. Hence the directional derivative can be obtained by simply taking the derivative of the equation with respect to a chosen particular direction m;

$$\frac{\partial u}{\partial m}(x) = \int_\Gamma \left(u\frac{\partial \Lambda}{\partial m} - \frac{\partial \lambda}{\partial m}\frac{\partial u}{\partial n_y}\right) ds_y\,, \quad x \in \Omega. \tag{1.5}$$

With this motivation,in this paper, we give an alternative representation for the nonlinear BIE (1.4). This simple method ,to be discussed in Section 2, is obtaining an analogue BIE for the equation (1.4) obtained by the equation (1.5). Using the standard collocation, we solve the reduced integral equation and the boundary integral equation (1.4). Throughout the experiment we show better performance in the computational efficiency and the rate of convergence than the result using the PCM [6]. This paper is organized as follows. In Section 2 we reduce a boundary integral equation from equation (1.5).

Section 3 contains a brief discussion on a numerical scheme. Finally in Section 4, we give computational results for some test problems.

2. Boundary Integral Equation

Let us first reduce boundary integral equation discussed in this paper. Note that the equation

$$\int_\Gamma \Lambda(x-y)\, ds_y = 1\,, \quad x \in \Omega\,, \tag{2.1}$$

is easily obtained using the rigid body motion. It follows that the identity

$$\int_\Gamma \frac{\partial \Lambda}{\partial m}\, ds_y = 0\,, \qquad x \in \Omega\,. \tag{2.2}$$

holds. The equation (1.5) can then be rewritten as

$$\frac{\partial u}{\partial m}(x) = \int_\Gamma \left([u(y) - u(x)] \frac{\partial \Lambda}{\partial m} - \frac{\partial \lambda}{\partial m} \frac{\partial u}{\partial n_y} \right) ds_y\,, \quad x \in \Omega. \tag{2.3}$$

In order to obtain the boundary integral equation starting from (2.3) we assume that the domain of the problem under study can be represented as shown in figure 1.

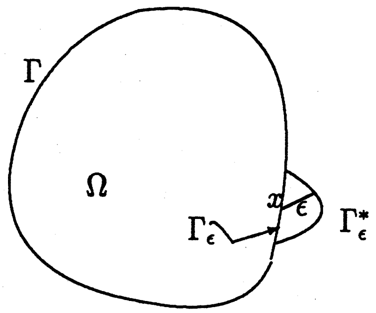

FIGURE 1. Body augmented by a circle centered at x.

Here, the internal point x is taken to a boundary point of the original domain, where initially the domain is considered argumented by part of a circle centered at x, with radius ϵ, whose boundary is defined by Γ_ϵ^*. The small part of Γ which has been removed by this hypothesis is now assumed to coincide with the outward normal directions n_x of the original boundary. The equation (2.3) is therefore written in the following form

(2.4)
$$\begin{aligned}\frac{\partial u(x)}{\partial n_x} &= \lim_{\epsilon\to 0}\int_{\Gamma-\Gamma_\epsilon+\Gamma_\epsilon^*}\left[(u(y)-u(x))\frac{\partial\Lambda}{\partial n_x}-\frac{\partial\lambda}{\partial n_x}\frac{\partial u}{\partial n_y}\right]ds_y\\ &= I_1+I_2+I_3\end{aligned}$$

where

$$\begin{aligned}I_1 &= \lim_{\epsilon\to 0}\int_{\Gamma_\epsilon^*}(u(y)-u(x))\frac{\partial\Lambda(x-y)}{\partial n_x}\,ds_y\,,\\ I_2 &= -\lim_{\epsilon\to 0}\int_{\Gamma_\epsilon^*}\frac{\partial u}{\partial n_y}\frac{\partial\lambda(x-y)}{\partial n_x}\,ds_y\,,\\ I_3 &= \lim_{\epsilon\to 0}\int_{\Gamma-\Gamma_\epsilon}\left[(u(y)-u(x))\frac{\partial\Lambda}{\partial n_x}-\frac{\partial\lambda}{\partial n_x}\frac{\partial u}{\partial n_y}\right]ds_y\,.\end{aligned}$$

The limit of I_1 and I_2 can be easily studied if integration is carried out in terms of a cylindrical coordinate system (r,θ), based at x. To estimate I_1 expand the potential $u(y)$ as Taylor's series about x, such that

$$u(y)-u(x)=\nabla u(x)\cdot(y-x)+O(|y-x|^2)\,,$$

or, if we simplify the notation and neglect the higher order terms (these will vanish as $\epsilon\to 0$) ;

$$u(y)-u(x)=\nabla u(x)\cdot(y-x)\,. \tag{2.5}$$

It follows that

$$
\begin{aligned}
I_1 &= \lim_{\epsilon\to 0}\frac{1}{2\pi}\int_{\Gamma_\epsilon^*}\nabla u(x)\cdot(y-x)\frac{n_x\cdot n_y}{|x-y|^2}\,ds_y \\
&= \frac{1}{2\pi}\frac{\partial u(x)}{\partial n_x}\lim_{\epsilon\to 0}\int_{\Gamma_\epsilon^*}\frac{n_y\cdot(y-x)}{|y-x|^2}\,ds_y \qquad (2.6)\\
&= \frac{1}{2\pi}\frac{\partial u(x)}{\partial n_x}\lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_{\Gamma_\epsilon^*}ds_y \quad \left(n_y=\frac{y-x}{|y-x|}\right)\\
&= \frac{1}{2}\frac{\partial u(x)}{\partial n_x}.
\end{aligned}
$$

In order to estimate the term I_2, we assume that u is a $C^{1,\alpha}(\Gamma)$ function, $\alpha\in(0,1)$. It follow that

$$
\begin{aligned}
I_2 &= \lim_{\epsilon\to 0}\int_{\Gamma_\epsilon^*}\frac{\partial u(y)}{\partial n_y}\left(\frac{(n_x-n_y)\cdot(y-x)}{|x-y|^2}+\Lambda(x-y)\right)ds_y \\
&= \frac{1}{2\pi}\frac{\partial u(x)}{\partial n_x}\lim_{\epsilon\to 0}\int_{\Gamma_\epsilon^*}\frac{n_x\cdot(y-x)}{|x-y|^2}\,ds_y \qquad (2.7)\\
&= \frac{1}{2\pi}\frac{\partial u(x)}{\partial n_x}\lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_{\Gamma_\epsilon^*}n_x\cdot n_y\,ds_y \quad \left(n_y=\frac{y-x}{|y-x|}\right)\\
&= \frac{1}{\pi}\frac{\partial u(x)}{\partial n_x}.
\end{aligned}
$$

Collecting identities (2.6) and (2.7), the boundary integral equation

$$
\beta\frac{\partial u(x)}{\partial n_x}+\fint_\Gamma\frac{\partial\lambda}{\partial n_x}\frac{\partial u}{\partial n_y}\,ds_y=\fint_\Gamma(u(y)-u(x))\frac{\partial\Lambda}{\partial n_x}\,ds_y \qquad (2.8)
$$

holds, where $\beta=1/2-1/\pi$. Substituting now the boundary condition (1.2) into (2.8), the boundary integral equation is obtained

(2.9)

$$
\begin{aligned}
\beta z(x)+\fint_\Gamma\frac{\partial\lambda}{\partial n_x}z(y)\,ds_y &= \fint_\Gamma(u(x)-u(y))\frac{\partial\Lambda}{\partial n_x}\,ds_y+\beta f(x)\\
&\quad+\fint_\Gamma\frac{\partial\lambda}{\partial n_x}f(y)\,ds_y
\end{aligned}
$$

where the function z is defined by $z(x)=g(x,u(x))$.

3. Collocation Method

At the present paper, we propose the collocation method to obtain the approximate solution of the problem. Let the smooth boundary Γ be parameterized by a smooth function $\gamma : [0, 2\pi] \to \Gamma$ which is 2π-periodic, and assume that $|\gamma'| > 0$. For any natural number n, let

$$\pi_n : 0 = t_0 < t_1 < \cdots < t_n = 2\pi$$

be a quasi-uniform mesh, $h = 2\pi/n$. Let Σ_h^d denote the space of 2π-periodic, $(d-1)$-times continuously differentiable smooth splines of degree d on π_n. The boundary element spaces in which we seek collocation approximations is denoted by S_h^d and is defined as

$$S_h^d := \{\phi(\gamma(\cdot)) : \phi \in \Sigma_h^d\}.$$

Let $\phi_{1,n}, \cdots, \phi_{n,n}$ be basis functions of the finite dimensional space Σ_h^d. Consider now approximate functions z_n and u_n given by

$$\begin{aligned} u_n(t) &= \sum_{j=1}^{n} b_{j,n}\phi_{j,n}(\gamma(t)), \quad t \in [0, 2\pi] \\ z_n(t) &= \sum_{j=n+1}^{2n} b_{j,n}\phi_{j,n}(\gamma(t)), \quad t \in [0, 2\pi]. \end{aligned} \tag{3.1}$$

To determine the unknown coefficients $b_{1,n}, \cdots, b_{2n,n}$, we require that u_n and z_n satisfy simultaneously equations (1.4) and (2.9) at the collocation points $t_i \in [0, 2\pi]$. Hence, we obtain a system of $2n$ linear algebraic equations in $b_{1,n}, \cdots, b_{2n,n}$:

$$\sum_{j=1}^{n} b_{j,n}[\frac{1}{2}\phi_{j,n}(\gamma(t_i)) - d_{ij}] - \sum_{j=n+1}^{2n} b_{j,n}d_{ij} = p_i, \tag{3.2}$$

and

$$\sum_{j=1}^{n} b_{j,n} c_{ij} - \sum_{j=n+1}^{2n} b_{i,n}[\beta\phi_{j,n}(\gamma(t_i)) + c_{ij}] = q_i \tag{3.3}$$

where

$$\begin{aligned}
d_{ij} &= \int_\Gamma \phi_{j,n}\Lambda(\gamma(t_i))\, ds_y\,, \quad 1 \le j \le n \\
&= \int_\Gamma \phi_{j,n}\lambda(\gamma(t_i))\, ds_y\,, \quad n+1 \le j \le 2n \\
c_{ij} &= \int_\Gamma (\phi_{j,n}(\gamma(t_i)) - \phi_{j,n}(y))\, \frac{\partial \Lambda}{\partial n_x}(\gamma(t_i) - y)\, ds_y\,, \quad 1 \le j \le n \\
&= \int_\Gamma \phi_{j,n} \frac{\partial \lambda}{\partial n_x}(\gamma(t_i) - y)\, ds_y\,, \quad n+1 \le j \le 2n \\
p_i &= -\int_\Gamma f\lambda(\gamma(t_i) - y)\, ds_y \quad \text{and} \\
q_i &= -\left[\beta f(\gamma(t_i)) + \int_\Gamma f \frac{\partial \lambda}{\partial n_x}(\gamma(t_i) - y)\, ds_y\right].
\end{aligned}$$

From the system of linear algebraic equations (3.2) and (3.3), we can determine the values of $b_{j,n}$ in (3.1). After the determination of the approximate values of $b_{j,n}$ from (3.2) and (3.3) we can proceed to the approximation of the solution of the problem from the equation (1.4). Consider now the approximate solution u_n^* to u given by

$$u_n^*(t) = \sum_{j=n}^{n} a_{j,n}\phi_{j,n}(\gamma(t_i))\,, \quad t \in [0.2\pi] \tag{3.4}$$

To determine the unknown coefficients $a_{j,n}$, we require that u_n^* satisfies the equation

$$\frac{1}{2}u_n^*(x) - \int_\Gamma u_n^*(y)\Lambda(x-y)\, ds_y = \int_\Gamma (g(y, u_n(y)) - f(y))\, \lambda(x-y)\, ds_y$$

at the collocation points $t_i \in [0, 2\pi]$. Hence, we obtain a system of n linear algebraic equations in $a_{1,n}, \cdots, a_{n,n}$:

$$\sum_{j=1}^{n} a_{j,n} \left[\frac{1}{2} \phi_{j,n}(\gamma(t_i)) - l_{ij} \right] = w_i \tag{3.5}$$

where

$$\ell_{ij} = \int_\Gamma \phi_{j,n} \Lambda(\gamma(t_i) - y)\, ds_y$$
$$w_i = \int_\Gamma (g(y, u_n(y)) - f(y))\, \lambda(\gamma(t_i) - y)\, ds_y\,.$$

These are the collocation methods for the numerical solution of the problem. Of course, several variation of these method are also possible.

4. Numerical Examples

Take the domain and its boundary as

$$\Omega = \{x \in R^2 \,|\, |x| < 0.8\},$$
$$\Gamma = \{x \in R^2 \,|\, |x| = 0.8\}.$$

We consider the following problems :

$$\Delta u(x) = 0,\ x \in \Omega, \tag{4.1}$$
$$\frac{\partial u(x)}{\partial n_x} = -g(x, u(x)) + f(x),\ x \in \Gamma, \tag{4.2}$$

where

(i) $g(x, u(x)) = u^4(x)$,
(ii) $g(x, u(x)) = |u^3(x)|$, or
(iii) $g(x, u(x)) = \sin(u(x))$.

The nonlinear function in (i) is typical in heat transfer and heat radiation problems.

For our numerical computation, in problems (i) and (ii), we choose f in (4.2) such that $u(x_1, x_2) = x_1$ is a solution of (4.1), (4.2). For the problem (iii), we choose f such that $u(x_1, x_2) = x_1^2 - x_2^2$.

Using the standard collocation method, our computation is carried out with the simple piecewise linear periodic splines. To discretize integrals in (3.3) and (3.4), let us take the boundary Γ to be represented by n isoparametric linear elements $e_1, e_2, \cdots, e_n$. A system of simultaneous equations in terms of nodal values of given and unknown data on the boundary may be obtained by taking the argument n of the free term in equation (3.3) and (3.4) to be located at each of the 2 nodes x_1, x_2, in turn, and substiting parametric interpolates.

Due to logarithmic singularity, the single–layer integral is discretized using the product integration rule [5] if there is a singularity and otherwise using the three point Trapezoidal rule. The other integrals is simply calculated in the given domain. Then the resulting linear algebraic equations are solved by the Gauss-elimination algorithm.

After determination of u_n from (3.3) and (3.4), we obtain an approximate solution u_n^* for real solution u using iteration for the equation (3.5). We iterated until the relative correction satisfied, for sufficiently small ϵ,

$$\frac{\|u_n^l - u_n^{l-1}\|_\infty}{\|u_n^l\|_\infty} \leq \epsilon.$$

where u_n^l is an approximate solution obtained from lth iteration.

Numerical results are given in the following Table 1 for solving (1.1)–(1.2) with the given function g. In Table 1, the column $\|u - u_h\|_\infty$ is the maximum error at the collocation points on $\partial\Omega$, the column n is the collocation number and the column EOC gives the empirically observed the expected order of convergence. we note that the experimental results are better than those obtained by the direct use of PCM [6].

In the figure, we draw the rate of convergence for $\|u - u_n\|_\infty$ and the collocation number n which is performed by the log–scaling with base 10.

n	$\|u-u_h\|_\infty$	EOC	$\|u-u_h\|_\infty$	EOC	$\|u-u_h\|_\infty$	EOC
10	3.5573E-2	–	3.2415E-2	–	1.3244E-1	–
20	8.4851E-3	2.0678	7.5988E-3	2.0929	4.9021E-2	1.4339
30	3.1912E-3	2.4118	2.8446E-3	2.4233	2.5249E-2	1.6363
40	1.4173E-3	2.8213	1.2671E-3	2.8110	1.5485E-2	1.6995
50	6.5508E-4	3.4585	5.9244E-4	3.4069	1.0552E-2	1.7189
60	2.9408E-4	4.3928	2.5826E-4	4.5540	7.7054E-3	1.7244
70	8.8840E-5	7.7652	7.6683E-5	7.8769	5.9027E-3	1.7288
150	–	–	–	–	1.5516E-3	–
200	–	–	–	–	9.4218E-4	–

TABLE 1. Experiment Results.

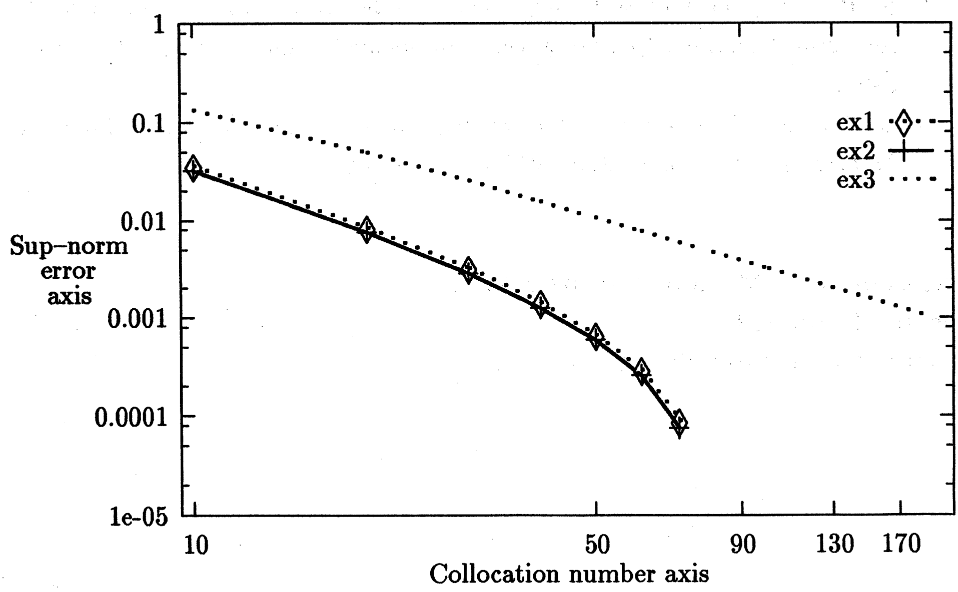

FIGURE 2. The error and collocation numbers with the given g.

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