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The Characterization of Optimal Control Using Delay Differential Operator

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ABSTRACT. In this paper we are concerned with optimal control problems whose costs are quadratic and whose states are governed by linear delay differential equations and general boundary conditions. The basic new idea of this paper is to introduce a new class of linear operators in such a way that the state equation subject to a starting function can be viewed as an inhomogeneous boundary value problem in the new linear operator equation. In this way we avoid the usual semigroup theory treatment to the problem and use only linear operator theory.

1. Introduction

Let $\Re$ be the field of all real numbers and let $\Re^n$ be the Euclidean real Hilbert Space of finite dimension $n(n \geq 1$ integer). For given $0 < \tau < t_1 < \infty$, $[-\tau, t_1]$ be an compact interval. Also for given integer p, let $\aleph_p$ denote the Hilbert Space of $x : [-\tau, t_1] \to \Re^p$ such that

$$|x| = \left(\int_{-\tau}^{t_1} x^*(t)x(t)dt \right)^{\frac{1}{2}} < \infty.$$

The inner product $< \cdot, \cdot >$ of $\aleph_p$ is denote by $< x, y >= \int_{-\tau}^{t_1} x^*(t)y(t)dt$. For $\alpha \in \Re^n$, $\|\alpha\| = (\alpha^*\alpha)^{\frac{1}{2}}$. Let $-\infty < a < b < \infty$ be real numbers. $\mathbf{L}_2^n[a, b]$ will denote the space of equivalence class of all square integrable functions from $[a, b]$ into $\Re^n$. $x_{[a,b]}$ denote the restriction of x to $[a, b]$.

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Consider a linear delay differential equation

$$
\begin{aligned}
&\dot{x}(t) = A_1(t)x(t) + A_2(t)x(t-\tau) + f(t), && t \in [0, t_1] \\
(1.1) \qquad &x(t) = f(t), && t \in [-\tau, 0]
\end{aligned}
$$

where $A_1(t)$, $A_2(t)$ are $n \times n$ real matrix valued functions whose columes are in $\aleph_n$ and $f \in \aleph_n$. Equation (1.1) is the simple delay-differential equation. However, analogous properties to those listed below can be derived for more general types of equations having time-varing delay, multiple delays, and so on.

It is well known that there exists a unique real continuous solution which satisfy (1.1) a.e on $[-\tau, t_1]$ when the given initial function $x(t) = f(t)$ is continuous on $[-\tau, 0]$. In Halany [5] he discussed about the solution of equation (1.1) when the initial function is continous, and found the solution in terms of fundamental matrix solution utilizing the adjoint systems. But our equation is different from [5] in that initial function is in $\mathbf{L}_2^n\left([-\tau, 0]; \Re^n\right)$.

In section 2 we introduce a new linear operator in such a way that the state equation subject to a starting function can be viewed as an inhomogenious boundary problem, and derive the adjoint operator of new operator, and then define the formal adjoint operator which will be play an important role in the characterization of the optimal control. Also, we discuss about fundamental matrix solution of (1.1) and adjoint system, and find the relation between two matrices. And then we discuss the solution of delay operator equations. Also we characterize the fundamental matrix which will be useful in practice.

In section 3 we consider the optimal control over a closed convex subset of $\mathbf{L}_2^m[-\tau, t_1]$ and develop the necessary and sufficient conditions for an optimal response pair in terms of adjoint equations and inclusions.

2. Delay differential operator

Define the delay differential operator $\Im : Dom\Im \to \aleph_n$ by

$$(\Im x)(t) = \begin{cases} \dot{x}(t) - A_1(t)x(t) - A_2(t)x(t-\tau), & t \in [0, t_1] \\ x(t), & t \in [-\tau, 0) \end{cases}$$

where

i) $Dom\Im = \{x \in \aleph_n | x_{[0,t_1]} \in \mathbf{AC}[0,t_1], \dot{x}_{[0,t_1]} \in \mathbf{L}_2^n[0,t_1]\}$

ii)$A_1(t)$ and $A_2(t)$ are $n \times n$ real matrix valued functions whose columes are in $\aleph_n$.

We see that $\Im$ is a linear operator.

THEOREM 2.1. *Let $\Im$ be difined as in the above. Then the adjoint operator $\Im^* : Dom\Im^* \to \aleph_n$ is*

$$(\Im^* y)(t) = \begin{cases} y(t) - A_2^*(t+\tau)y(t+\tau), & t \in [-\tau, 0) \\ -y(t) - A_1^*(t)y(t) - A_2^*(t+\tau)y(t+\tau), & t \in [0, t_1 - \tau) \\ -y(t) - A_1^*(t)y(t), & t \in [t_1 - \tau, t_1], \end{cases}$$

where

$$Dom\Im^* = \{y \in \aleph_n \mid y(t_1) = y(0) = 0, y_{[0,t_1]} \in \mathbf{AC}[0,t_1], \dot{y}_{[0,t_1]} \in \mathbf{L}_2^n[0,t_1]\}.$$

Now let's define the formal adjoint operator, which will be useful to the characterization of optimal control, $\Im^+ : Dom\Im^+ \to \aleph_n$ by

$$(\Im^+ y)(t) = \begin{cases} y(t) - A_2^*(t+\tau)y(t+\tau), & t \in [-\tau, 0) \\ -y(t) - A_1^*(t)y(t) - A_2^*(t+\tau)y(t+\tau), & t \in [0, t_1 - \tau) \\ -y(t) - A_1^*(t)y(t), & t \in [t_1 - \tau, t_1] \end{cases}$$

where

$$Dom\Im^+ = \left\{ y \in \aleph_n | y_{[0,t_1]} \in \mathbf{AC}[0,t_1], \dot{y}_{[0,t_1]} \in \mathbf{L}_2^n[0,t_1] \right\}.$$

Note that $\Im^* \subset \Im^+$.

Define a $n \times n$ matrix valued functions $X(t,s)$ and $Y(s,t)$ on $\Re \times \Re$ as follows; for each $s \in \Re$ fixed,

$$\text{(2-1)} \quad \begin{aligned} \text{i) } & X(t,t) = I_n \\ & X(t,s) = 0, \qquad t > s \\ & \frac{\partial}{\partial t} X(t,s) = A_1^*(t)X(t,s) + A_2^*(t)Y(t-\tau,s), \; t \geq s, \end{aligned}$$

$$\begin{aligned} \text{ii) } & Y(t,t) = I_n \\ & Y(s,t) = 0 \qquad s > t \\ & \frac{\partial}{\partial s} Y(s,t) = A_1^*(s)Y(s,t) + A_2^*(s+\tau)Y(s+\tau,t), \; s \leq t. \end{aligned}$$

Then we have the following theorem.

THEOREM 2.2. $X(t,s) = Y^*(s,t)$, *for all* $t, s \in \Re$.

The following theorem characterize the fundamental matrix solution $X(t,s)$ of (2.1) in terms of the fundamental matrix solution $\Phi(t)$ of $\dot{x}(t) = A_1(t)x(t)$.

THEOREM 2.3. *Let* $\Phi(t)$ *be the* $n \times n$ *fundamental matrix solution of* $\dot{x}(t) = A_1(t)x(t)$. *Then*

$$X(t,s) = \Phi(t) \sum_{i=0}^{\ell} H_i(t,s)\Phi^{-1}(s), \; t > s$$

where
i) $\ell \in N$ *such that* $t \in [s + \ell\tau, s + (\ell + 1)\tau]$
ii) $H_0(t, s) = I$.
iii) for $j = 1, 2, \cdots, \ell$

$$H_j(t,s) = \begin{cases} \int_{s+(j-1)\tau}^{t-\tau} h(\alpha)H_{j-1}(\alpha,s)d\alpha, & s + j\tau \le t \le s + (j+1)\tau \\ 0, & otherwise \end{cases}$$

here $h(\alpha) = \Phi^{-1}(\alpha)A_2(\alpha + \tau)\Phi(\alpha + \tau)$.

PROOF. We construct $X(t, s)$ using "step by step"method. Since for $t < s + \tau$, $X(t - \tau, s) = 0$, for $s \le t < s + \tau$, (2-1) becomes

$$\frac{\partial}{\partial t}X(t,s) = A_1(t)X(t,s),$$
$$X(s,s) = I.$$

Now $X(t, s)$ is the matrix solution of $\dot{x}(t) = A_1(t)x(t)$.
Thus $X(t, s) = \Phi(t)C(s)$. But $X(s, s) = I$. Therefore $C(s) = \Phi^{-1}(s)$.
That is,

$$X(t,s) = \Phi(t)\Phi^{-1}(s), \quad \text{for } s \le t < s + \tau.$$

For $s + \tau \le t \le s + 2\tau$,

$$\frac{\partial}{\partial t}X(t,s) = A_1(t)X(t,s) + A_2(t)X(t-\tau,s)$$

with $X(s + \tau, s) = \Phi(s + \tau)\Phi^{-1}(s)$. Therefore

$$\begin{aligned} X(t,s) &= \Phi(t)\Phi^{-1}(s+\tau)X(s+\tau,s) \\ &\quad + \int_{s+\tau}^{t} \Phi(t)\Phi^{-1}(\alpha)A_2(\alpha)X(\alpha-\tau,s)d\alpha \\ &= \Phi(t)\Phi^{-1}(s+\tau)\Phi(s+\tau)\Phi^{-1}(s) \\ &\quad + \int_{s}^{t-\tau} \Phi(t)\Phi^{-1}(\alpha+\tau)A_2(\alpha+\tau)\Phi(\alpha)\Phi^{-1}(s)d\alpha \\ &= \Phi(t)\left(I + \int_{s}^{t-\tau} \Phi^{-1}(\alpha+\tau)A_2(\alpha+\tau)\Phi(\alpha)d\alpha\right)\Phi^{-1}(s). \end{aligned}$$

Let

$$H_1(t,s) \equiv \int_s^{t-\tau} \Phi^{-1}(\alpha+\tau)A_2(\alpha+\tau)\Phi(\alpha)d\alpha.$$

Then $X(t,s) = \Phi(t)(I + H_1(t,s))\Phi^{-1}(s)$.
Now we show "by induction "that for $s+\ell\tau \le t \le s+(\ell+1)\tau$, where

i) $H_0(t,s) = I$

ii) $H_i(t,s) = \displaystyle\int_{s+(i-1)\tau}^{t-\tau} h(\alpha)H_{i-1}(\alpha,s)d\alpha, \quad i = 1,2,\cdots,\ell$

here $h(\alpha) = \Phi^{-1}(\alpha+\tau)A_2(\alpha+\tau)\Phi(\alpha)$.
Suppose that for $s+(\ell-1)\tau \le t \le s+\ell\tau$,

$$X(t,s) = \Phi(t)\sum_{i=0}^{\ell-1} H_i(t,s)\Phi^{-1}(s),$$

Then for $s+\ell\tau \le t \le s+(\ell+1)\tau$,

$$\begin{aligned}
X(t,s) &= \Phi(t)\Phi^{-1}(s+\ell\tau)X(s+\ell\tau,s)\\
&\qquad + \int_{s+\ell\tau}^{t} \Phi(t)\Phi^{-1}(\alpha)A_2(\alpha)X(\alpha-\tau,s)d\alpha\\
&= \Phi(t)\Phi^{-1}(s+\ell\tau)\Phi(s+\ell\tau)\sum_{i=0}^{\ell-1} H_i(s+\ell\tau,s)\Phi^{-1}(s)\\
&+ \Phi(t)\int_{s+(\ell-1)\tau}^{t-\tau} \Phi^{-1}(\alpha+\tau)A_2(\alpha+\tau)\Phi(\alpha)\sum_{i=0}^{\ell-1} H_1(\alpha,s)\Phi^{-1}(s)d\alpha\\
&= \Phi(t)\Bigg[I + \left(\int_s^{s+(\ell-1)\tau} h(\alpha)d\alpha + \int_{s+(\ell-1)\tau}^{t-\tau} h(\alpha)d\alpha\right)\\
&+ \left(\int_{s+\tau}^{s+(\ell-1)\tau} h(\alpha)H_1(\alpha,s)d\alpha + \int_{s+(\ell-1)\tau}^{t-\tau} h(\alpha)H_1(\alpha,s)d\alpha\right)
\end{aligned}$$

$$\dots$$

$$+ \left(\int_{s+(\ell-2)\tau}^{s+(\ell-1)\tau} h(\alpha)H_{\ell-2}(\alpha, s)d\alpha + \int_{s+(\ell-1)\tau}^{t-\tau} h(\alpha)H_{\ell-2}(\alpha, s)d\alpha \right)$$

$$\left. + \int_{s+(\ell-1)\tau}^{t-\tau} h(\alpha)H_{\ell-1}(\alpha, s)d\alpha \right] \Phi^{-1}(s), \ t > s.$$

Thus we have

$$X(t, s) = \Phi(t) \sum_{i=0}^{\ell} H_i(t, s)\Phi^{-1}(s).$$

In the following theorem we state the solution of delay differential operator equation.

THEOREM 2.4. *Let* $f \in \mathbf{L}_2^n[-\tau, t_1]$. *Then* $(\Im x)(t) = f(t)$, *for a.a.* $t \in [-\tau, t_1]$ *if and only if*

$$i) \ x(t) = f(t), \quad for \ a.a. t \in [-\tau, t_1)$$

$$ii) \ x(t) = X(t, 0)x_0 + \int_{-\tau}^{0} X(t, s+\tau)A_2(s+\tau)x(s)ds + \int_0^t X(t, s)f(s)ds, \ t \in [0, t_1]$$

where $x_0 \in \Re^n$ *is given.*

Now we state the corollary which will be useful to characterize the optimal control.

COROLLARY 2.5. *Let* $f \in \mathbf{L}_2^n[0, t_1+\tau]$. *Then* $(\Im^+ y)(t) = f(t)$, $t \in [-\tau, t_1]$ *if and only if*

$$y(t) = \begin{cases} A_2^*(t+\tau)y(t+\tau) + f(t), & t \in [-\tau, 0] \\ Y(t_1, t)\beta + \int_t^{t_1} Y(s, t)f(s)ds, & t \in [0, t_1] \end{cases}$$

where $\beta \in \Re^n$ *is given.*

3. Optimal control over $\mathbf{L}_2^m[0,t_1]$

For $i=1,2$, let $\mathbf{F}_i : \mathbf{L}_2^m[0,t_1] \otimes \mathbf{L}_2^n[0,t_1] \to \Re^d$ be defined by

$$\mathbf{F}_i(u,x) \equiv \int_0^{t_1} (f_{i1}^*(t)x(t) + f_{i2}^*(t)u(t))dt$$

where f_{i1}, f_{i2} are $n \times d$, $m \times d$ real valued matrices whose columnes are in $\mathbf{L}_2^n[0,t_1]$, $\mathbf{L}_2^m[0,t_1]$, respectively. Let

$$\mathbf{J}(u,x) \equiv \int_0^{t_1} (|Uu|^2 + |Wx|^2)dt + |F_1(u,x)|^2,$$

for $u \in \mathbf{L}_2^m[0,t_1]$, $x \in Dom\Im$, where $|\cdot|$ is the Euclidean norm.

Let $\gamma \in \Re^d$ be given. We consider the following optimal control problem;

Minimize $\mathbf{J}$ over all $\{u,x\}$ such that

$$\begin{aligned} &\text{i) } u \in \mathbf{L}_2^m[0,t_1],\ x \in Dom\Im \\ &\text{ii) } x(t) = \phi(t),\ t \in [-\tau,0) \\ &\quad (\Im x)(t) = B(t)u(t),\ t \in [-\tau,t_1] \\ &\text{iii) } F_2(u,x) = \gamma, \end{aligned}$$

here $B(t)$ is a $n \times m$ real-valued matrix whose columnes are in $\mathbf{L}_2^m[0,t_1]$ and $\phi(t) \in \mathbf{L}_2^n[0,t_1]$.

If $\mathbf{F}_2(u,x) = Mx(0)+Nx(t_1)$ for some constant matrices M, N and $\mathbf{F}_1(u,x) = 0$, the above problem becomes a classical control problem associated with two point boundary condition.

Let $\mathbf{D} \equiv \{\{u,x\} | u,x \text{ satisfy i),ii) and iii)}\}$. An element $\{u,x\} \in \mathbf{D}$ is called a succesful control response pair. It is possible that $\mathbf{D}$ is empty. When $\{u^+,x^+\}$ minimize $\mathbf{J}$ over $\mathbf{D}$, it is called an optimal-response pair.

Define $\mathbf{T} : \mathbf{L}_2^m[0, t_1] \rightarrow \mathbf{L}_2^n[0, t_1]$ by

$$\mathbf{T}(u) = \int_0^t X(t,s)B(s)u(s)ds$$

and $\mathbf{T} : \mathbf{L}_2^m[0, t_1] \otimes \Re_n \rightarrow \mathbf{L}_2^n[0, t_1]$ by

(3-1) $$\widetilde{\mathbf{T}}(u, x_0) \equiv X(t, 0)x_0 + \mathbf{T}(u), \ t \in [0, t_1]$$

Then the state equation which satisfy condition ii) is expressed by

(3-2) $$x(t) = \widetilde{\mathbf{T}}(u, x_0) + h(t), t \in [0, t_1].$$

Here $h(t) = \int_{-\tau}^{0} X(t, s+\tau)A_2(s+\tau)\phi(s)ds, t \in [0, t_1]$.

Let's define the following ; for i=1,2,

$$Q_i = \int_0^t f_{i1}X(t,0)dt,$$

$$m_i(t) = \int_t^{t_1} f_{i1}^*(s)X(s,t)dsB(t) + f_{i2}^*(t), \ t \in [0, t_1]$$

and $\gamma_i = \int_0^{t1} f_{i1}^*(t)h(t)dt$.

Also define $\mathbf{M}_i : \mathbf{L}_2^m[0, t_1] \rightarrow \mathbf{R}^d$ by

$$\mathbf{M}_i(u) = \int_0^{t_1} m_i(t)u(t)dt$$

Then for i=1,2,

$$\mathbf{F}_i(u, x) = \mathbf{Q}_i x_0 + \mathbf{M}_i(u) + \gamma_i$$

Also, for i=1,2, define $\mathbf{N}_i : \Re^n \otimes \mathbf{L}_2^m[0, t_1] \rightarrow \Re_d$ by

$$\mathbf{N}_i(x_0, u) = \mathbf{Q}_i x_0 + \mathbf{M}_i(u).$$

Then $\mathbf{F}_2(u,x)=\gamma$ becomes

$$\mathbf{N}_2(x_0,u)=\gamma-\gamma_2.$$

Therefore $\mathbf{D}$ is not empty if and only if $\gamma-\gamma_2\in Range(\mathbf{N}_2)$. Through this section, we assume that $\mathbf{D}$ is non-empty. Let $\{\widetilde{x}_0,\widetilde{u}\}$ be an arbitrary but fixed elementary of $\mathbf{N}_2^{-1}(\gamma-\gamma_2)$. Then

$$\mathbf{N}_2^{-1}(\gamma-\gamma_2) \;=\; \{\widetilde{x}_0,\widetilde{u}\} \;+\; Null(\mathbf{N}_2). \tag{3-3}$$

Thus

$$\mathbf{D}=\left\{\left\{\widetilde{u}+\hat{u},\widetilde{\mathbf{T}}(\widetilde{u}+\hat{u},\widetilde{x}_0+\hat{x}_0)+h\right\}|\{\hat{x}_0,\hat{u}\}\in Null\mathbf{N}_2\right\}.$$

Let

$$\mathbf{E}=\{u\in\mathbf{L}_2^m[0,t_1]|\gamma-\gamma_2-\mathbf{M}_2(u)\in Range\mathbf{Q}_2\}$$

and pick an arbitrary but fixed algebraic operator part of $\mathbf{Q}_2$, say $\mathbf{Q}_2^\sharp$. Then $\mathbf{N}_2\,(\hat{x}_0,\hat{u})=0$ if and only if $\hat{u}\in\mathbf{E}$ and $\hat{x}_0=-\mathbf{Q}_2^\sharp\mathbf{M}_2(\hat{u})+q,\ q\in Null\mathbf{Q}_2$. Therefore,

$$\mathbf{D}=\left\{\widetilde{u}+\hat{u},\widetilde{\mathbf{T}}\left(\widetilde{u}+\hat{u},\widetilde{x}_0-\mathbf{Q}_2^\sharp\mathbf{M}_2(\hat{u})+q\right)+h|\hat{u}\in\mathbf{E},q\in Null\ \mathbf{Q}_2\right\}$$

Note that

$$\mathbf{F}_1(u,x)=\mathbf{Q}_1\left(\widetilde{x}_0-\mathbf{Q}_2^\sharp\mathbf{M}_2(\hat{u})+q\right)+\mathbf{M}_1(\widetilde{u}+\hat{u}),$$

for $q\in Null\ \mathbf{Q}_2$ and $\hat{u}\in\mathbf{E}$.

Now

$$\begin{aligned}\mathbf{J}(u,x)&=\|\{Uu,Wx,\mathbf{F}_1(u,x)\}\|^2\\&=\|\Big\{U(\widetilde{u}+\hat{u}),W\left[\widetilde{\mathbf{T}}\left(\widetilde{u}+\hat{u},\widetilde{x}_0-\mathbf{Q}_2^\sharp\mathbf{M}_2(\hat{u})+q\right)+h\right],\\&\qquad\mathbf{Q}_1\left(\widetilde{x}_0-\mathbf{Q}_2^\sharp\mathbf{M}_2(\hat{u})+q\right)+\mathbf{M}_1(\widetilde{u}+\hat{u})\Big\}\|\\&=\|\left\{U\hat{u},W\widetilde{\mathbf{T}}\left(\hat{u},q-\mathbf{Q}_2^\sharp\mathbf{M}_2(\hat{u})\right),\mathbf{M}_1(\hat{u})-\mathbf{Q}_1\mathbf{Q}_2^\sharp\mathbf{M}_2(\hat{u})+\mathbf{Q}_1q\right\}\\&\qquad+\left\{U\widetilde{u},W\left[\widetilde{\mathbf{T}}(\widetilde{u},\widetilde{x}_0)+h\right],\mathbf{Q}_1\widetilde{x}_0+\mathbf{M}_1(\widetilde{u})\right\}\|\end{aligned}$$

for $q \in Null\mathbf{Q}_2$ and $\hat{u} \in \mathbf{E}$.

Thus we have the following lemma.

LEMMA 3.1. *$\{u^+, x^+\}$ is an optimal if and only if*

$$u^+ = \widetilde{u} + \hat{u}, \quad x = \widetilde{\mathbf{T}}\left(\widetilde{u} + \hat{u}, \widetilde{x}_0 + q - \mathbf{Q}_2^\sharp \mathbf{M}_2(\hat{u})\right) + h$$

for some $\hat{u} \in \mathbf{E}$ and $q \in Null\mathbf{Q}_2$ such that $\{\hat{u}, q\}$ minimize

$$\|\zeta + P(u^+) + \{0, W(0,q), \mathbf{Q}_1 q\}\|,$$

here

$$\zeta = \left\{U\widetilde{u}, W\left[\widetilde{\mathbf{T}}(\widetilde{u}_0, \widetilde{x}_0) + h\right], \mathbf{Q}_1\widetilde{x}_0 + \mathbf{M}_1(\widetilde{u})\right\}$$

and

$$P(\hat{u}) = \left\{U\hat{u}, W\widetilde{\mathbf{T}}\left(\hat{u}, q - \mathbf{Q}_2^\sharp \mathbf{M}_2(\hat{u})\right), \mathbf{M}_1(\hat{u}) - \mathbf{Q}_1\mathbf{Q}_2^\sharp \mathbf{M}_2(\hat{u})\right\}.$$

Let's define a closed linear relation $\mathbf{P}$ in $(\mathbf{L}_2^m \times \mathbf{R}^n) \times (\mathbf{L}_2^m \times \mathbf{L}_2^m \times \mathbf{R}^d)$ by

$$\mathbf{P} = \left\{\left\{\{\hat{u}, q\}, P(\hat{u}) + \{0, W\widetilde{\mathbf{T}}(0,q), \mathbf{Q}_1 q\}\right\} | \hat{u} \in \mathbf{E}, q \in Null\mathbf{Q}_2\right\}.$$

In terms of the relation $\mathbf{P}$, the above lemma can be restated in the following forms.

LEMMA 3.2. *The followings are equivalent.*

i) $\{u^+, x^+\}$*is an optimal.*

ii) $u^+ = \widetilde{u} + \hat{u}, \quad x^+ = \widetilde{\mathbf{T}}\left(\widetilde{u} + \hat{u}, \widetilde{x}_0 + q - \mathbf{Q}_2^\sharp \mathbf{M}_2(\hat{u})\right) + h$
for some $\{\hat{u}, q\} \in \mathbf{E} \times Null\mathbf{Q}_2$ which is an Least Square Solution of $-\zeta \in \mathbf{P}(\hat{u}, q)$.

iii) $u^+ = \widetilde{u} + \hat{u}, \quad x^+ = \widetilde{\mathbf{T}}\left(\widetilde{u} + \hat{u}, \widetilde{x}_0 + q - \mathbf{Q}_2^\sharp \mathbf{M}_2(\hat{u})\right) + h$
for some $\{\hat{u}, q\} \in \mathbf{E} \times Null\mathbf{Q}_2$ such that
$\zeta + \mathbf{P}(\hat{u}, q) \in (Range\mathbf{P})^\perp$.

We now state an existence theorem for an optimal control which follow easily from abstract setting from section 3 of Reference[6], applied to the relation $\mathbf{P}$.

THEOREM 3.3. *Assume that there exists a successful control-response pair. An optimal control exists if and only if*

$$-\zeta \in (Range\mathbf{P})^{\perp} \oplus Range\mathbf{P}.$$

The following theorem characterize an optimal pair in terms of integral inclusion and matrix equation.

THEOREM 3.4. *Let $\{u^+, x^+\} \in \mathbf{D}$. Then $\{u^+, x^+\}$ is an optimal pair if and only if there exists $\eta \in Dom\Im^+$ and $\delta_i \in Dom\Im^+$ $(i = 1, 2)$ columnwise such that*

$$\begin{aligned}
&i)\ (\Im^+\eta)(t) = W^*(t)W(t)x^+(t), \quad t \in [0, t_1]\\
&\quad \eta(t_1) = 0\\
&ii)\ \text{for } j = 1, 2, \cdots, d,\\
&\quad (\Im^+\delta_{ij})(t) = f_{i1_j}(t), \quad t \in [0, t_1]\\
&\quad \delta_{ij}(t_1) = 0\\
&iii)\ U^*Uu^+ + B^*\eta + (B^*\delta_1 + f_{12})\mathbf{F}_1(u^+, x^+)\\
&\quad - (B^*\delta_2 + f_{22})\left(\mathbf{Q}_2^\sharp\right)^*(X(\cdot, 0)W^*Wx^+ + \mathbf{Q}_1^*\mathbf{F}_1(u^+, x^+))\\
&\quad \in \mathbf{E}^{\perp}
\end{aligned}$$

and

$$X^*(\cdot, 0)W^*Wx^+ + \mathbf{Q}_1^*\mathbf{F}_1(u^+, x^+) \in (Null\ \mathbf{Q}_2)^{\perp}.$$

PROOF. By Lemma 3.2, $\{u^+, x^+\}$ is an optimal if and only if u^+ and x^+ have the same representation as in the lemma for some $\{\hat{u}, q\} \in (Range\mathbf{P})^{\perp}$ satisfying

$$\zeta + \mathbf{P}(\hat{u}, q) \in (Range\mathbf{P})^{\perp}. \tag{3-4}$$

Since

$$\zeta + \mathbf{P}(\hat{u}, q) = \{U\hat{u}, W\hat{x}, \mathbf{F}_1\left(u^+, x^+\right)\},$$

(3-4) becomes

$$\begin{aligned} &\langle Uu^+, U\hat{u}\rangle + \langle \mathbf{F}_1\left(u^+, x^+\right), \left(\mathbf{M}_1 - \mathbf{Q}_1\mathbf{Q}_2^\sharp\mathbf{M}_2(\hat{u})\right) + \mathbf{Q}_1 q\rangle \\ &\quad + \langle Wx^+, W\widetilde{\mathbf{T}}\left(\hat{u}, q - \mathbf{Q}_2^\sharp\mathbf{M}_2(\hat{u})\right)\rangle = 0, \text{for all } \hat{u} \in \mathbf{E}. \end{aligned} \tag{3-5}$$

Note that

$$\widetilde{\mathbf{T}}\left(\hat{u}, q - \mathbf{Q}_2^\sharp\mathbf{M}_2(\hat{u})\right) = Y(t,0)\left(q - \mathbf{Q}_2^\sharp\mathbf{M}_2\hat{u}\right) + (\mathbf{T}\hat{u})(t), \quad t \in [0, t_1].$$

Therefore (3-5) implies

$$\begin{aligned} &\langle U^*Uu^+ + \mathbf{T}^*W^*Wx^+ + \mathbf{M}_1^*\mathbf{F}_1\left(u^+, x^+\right) \\ &\quad - \mathbf{M}_2^*\left(\mathbf{Q}_2^\sharp\right)^*\left[Y^*(0,\cdot)W^*Wx^+ - \mathbf{Q}_1^*\mathbf{F}_1(u^+, x^+)\right], \hat{u}\rangle \\ &\quad + \langle Y^*(0,\cdot)W^*Wx^+ + \mathbf{Q}_1^*\mathbf{F}_1\left(u^+, x^+\right), q\rangle = 0, \end{aligned} \tag{3-6}$$

for all $\hat{u} \in \mathbf{E}$ and $q \in \mathit{Null}\ \mathbf{Q}_2$. That implies

$$\begin{aligned} &U^*Uu^+ - \mathbf{M}_2^*\left(\mathbf{Q}_2^\sharp\right)^*\left(Y^*(0,\cdot)W^*Wx^+ - \mathbf{Q}_1^*\mathbf{F}_1(u^+, x^+)\right) \\ &\quad + \mathbf{T}^*W^*Wx^+ + \mathbf{Q}_1^*\mathbf{F}_1\left(u^+, x^+\right) \in \mathbf{E}^\perp \end{aligned} \tag{3-7}$$

and

$$Y^*(0,\cdot)W^*Wx^+ + \mathbf{Q}_1^*\mathbf{F}_1\left(u^+, x^+\right) \in (\mathit{Null}\mathbf{Q}_2)^\perp.$$

Note that

$$\mathbf{T}^*w = B^*(t)\int_t^{t_1} x^*(t,s)w(s)ds$$

and

$$\mathbf{M}_1(v) = \left(f_{i2}(t) + B^*(t) \int_t^{t_1} X^*(t,s) f_{i1}(s) ds \right) v(t).$$

Let

$$\eta(t) = \int_t^{t_1} X^*(t,s) W^*(s) W(s) x^+(s) ds$$

and

$$\delta_i(t) = \int_t^{t_1} X^*(t,s) f_{i1}(s) ds.$$

Then (3-7) becomes

$$\begin{aligned} &U^*Uu^+ + B^*\eta + (B^*\delta_1 + f_{12})\mathbf{F}_1(u^+,x^+) \\ &- (B^*\delta_2 + f_{22})\left(\mathbf{Q}_2^\sharp\right)^* (X(\cdot,0)W^*Wx^+ + \mathbf{Q}_1^*\mathbf{F}_1(u^+,x^+)) \\ &\in \mathbf{E}^\perp \end{aligned}$$

This proves the theorem.

The above theorem covers a wide variety of optimal control problems of delay differential equation subject to generalized two-point boundary conditions with a fixed initial function. In the following we drive the optimal control of classical two point boundary problem.

COROLLARY 3.5. *Consider the problem of minimizing the functional*

$$\mathbf{J}_1(u,x) = \int_0^{t_1} \left(|Uu|^2 + |Wx|^2\right) dt$$

over all $\{u,x\}$ *such that*

1) $u \in \mathbf{L}_2^m[0,t_1]$, $x \in Dom\Im$

2) $x(t) = \phi(t)$, $t \in [-\tau, 0)$

$(\Im x)(t) = B(t)u(t)$, $t \in [0,t_1]$

3) $Mx(0) + Nx(t_1) = \gamma$,

where M *and* N *are* $d \times n$ *constant matrices.*

Then a succesful control-responce pair $\{u^+, x^+\}$ *is an optimal if and only if there exists* $\eta \in Dom\Im^+$ *such that*

$$i)\ (\Im^+\eta)(t) = W^*(t)W(t)x^+(t), \quad t \in [-\tau, t_1]$$
$$\eta(t_1) = 0$$
$$ii)\ U^*Uu^+ + B^*\eta - B^*(t)X^*(t,s)N^*\left(\mathbf{Q}_3^\sharp\right)^* X(\cdot,0)W^*Wx^+ \in (\mathbf{E}_1)^\perp$$

and

$$X^*(\cdot,0)W^*Wx^+ \in (Null\ \mathbf{Q}_3)^\perp.$$

Here $\mathbf{Q}_3 = M + NX(t_1, 0)$ *and*

$$\mathbf{E}_1 = \{u \in \mathbf{L}_2^m[0,t_1] | \gamma - \gamma_2 - N(\mathbf{T}u)(t_1) \in Range\ \mathbf{Q}_3\}$$

PROOF. We use Theorem 3.4 to prove this corollary. Since

$$\begin{aligned}\widetilde{\mathbf{F}}_2(u,x) &= \int_0^{t_1} [f_{21}^*(t)x(t) + f_{22}^*(t)u(t)]\, dt \\ &= Mx(0) + Nx(t_1)\end{aligned}$$

for all (u,x) satisfying $(\Im x)(t) = B(t)u(t)$ with $x(t) = \phi(t)$, $t \in [-\tau, 0)$,we have following relation;

$$\begin{aligned}&\int_0^{t_1} [f_{21}^*(t)\left(X(t,0)x(0) + (\mathbf{T}u)(t) + h(t)\right) + f_{22}^*(t)u(t)] \\ &\quad = Mx(0) + N\left(X(t_1,0)x(0) + (\mathbf{T}u)(t_1) + h(t_1)\right)\end{aligned}$$

for all $x(0) \in \Re_n$ and $u \in \mathbf{L}_2^m[0,t_1]$. Also

$$\int_{21}^{*} (t)h(t) = Nh(t_1) \equiv \gamma_2.$$

Hence

$$\mathbf{Q}_3 = M + NX(t_1, 0)$$

and

$$B^*(t)\delta_2 + f_{22}(t) = B^*(t)X^*(t,s)N^*.$$

Therefore, iii) of Theorem 3.4 becomes

$$U^*Uu^+ + B^*\eta - B^*(t)X^*(t,s)N^*\left(\mathbf{Q}_3^\sharp\right)^* X(\cdot,0)W^*Wx^+ \in (\mathbf{E}_1)^\perp.$$

COROLLARY 3.5. *Assume that $d = n$ and Null $\mathbf{Q}_2 = \{0\}$. Let $\{u^+, x^+\}$ is an optimal pair if and only if there exists $\eta \in Dom\Im^+$ and $\delta_i \in Dom\Im^+$ $(i = 1,2)$ columnwise such that*

i) $(\Im^+\eta)(t) = W^*(t)W(t)x^+(t), \quad t \in [-\tau, t_1]$

$\eta(t_1) = 0$

ii) for $j = 1, 2, \cdots, d$,

$(\Im^+\delta_{ij})(t) = f_{i1_j}(t), \quad t \in [-\tau, t_1]$

$\delta_{ij}(t_1) = 0$

iii) $U^*Uu^+ + B^*\eta + (B^*\delta_1 + f_{12})\mathbf{F}_1(u^+, x^+)$

$- (B^*\delta_2 + f_{22})\left(\mathbf{Q}_2^\sharp\right)^* \left(X(\cdot,0)W^*Wx^+ + \mathbf{Q}_1^*\mathbf{F}_1(u^+,x^+)\right) = 0, t \in [0, t_1]$

PROOF. Since *Null* $\mathbf{Q}_2 = \{0\}$, $\mathbf{E} = \mathbf{L}_2^m[-\tau, t_1]$. Therefore iii) of Theorem 3.4 becomes

$$U^*Uu^+ + B^*\eta + (B^*\delta_1 + f_{12})\mathbf{F}_1(u^+, x^+)$$
$$- (B^*\delta_2 + f_{22})\left(\mathbf{Q}_2^\sharp\right)^* \left(X(\cdot,0)W^*Wx^+ + \mathbf{Q}_1^*\mathbf{F}_1(u^+,x^+)\right) = 0, t \in [0, t_1]$$

and

$$X(\cdot,0)W^*Wx^+ + \mathbf{Q}_1^*\mathbf{F}_1\left(u^+,x^+\right) \in (Null\ \mathbf{Q}_2)^\perp$$

is always true. Thus we have the corollary.

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