JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume 7, June 1994

Strong Higher Derivations on Ultraprime Banach Algebras

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Abstract. In this paper we show that if $\{H_n\}$ is a continuous strong higher derivation of order n on an ultraprime Banach algebra with a constant c, then $c||H_1||^2 \leq 4||H_2||$ and for each $1 \leq l < n$

$$c^2||H_l||\ ||H_{n-l}|| \leq 6||H_n|| + \frac{3}{2} \sum_{\substack{i+j+k=n \\ i,j,k \geq 1}} ||H_i||\ ||H_j||\ ||H_k|| + \frac{3}{2} \sum_{\substack{i+k=n \\ i \neq l,\ n-l}} ||H_i||\ ||H_k||$$

and for a strong higher derivation $\{H_n\}$ of order n on a prime ring A we also show that if $[H_n(x), x] = 0$ for all $x \in A$ and for every $n \geq 1$, then A is commutative or $H_n = 0$ for every $n \geq 1$.

I. Introduction

Let A be a ring. For each $a,\ b \in A$, $M_{a,b} : A \to A$ is a mapping defined by $M_{a,b}(x) = axb$ for $x \in A$. Recall that A is *prime* if $M_{a,b} = 0$ implies $a = 0$ or $b = 0$ for each $a,\ b \in A$.

A complex normed algebra is *ultraprime* with a constant c if there exists a constant $c > 0$ such that for all $a,\ b \in A$

$$||M_{a,b}|| \geq c||a||\ ||b||$$

Note that every prime C^*-algebra is ultraprime with constant 1.

Received by the editors on June 27, 1994.
1980 *Mathematics subject classifications*: Primary 46H40.

A sequence $\{H_0, H_1, \cdots, H_n\}$ of linear operators on A is a *higher derivation* of order n if for each $m = 0, 1, 2, \cdots, n$ and any $x, y \in A$ the operator H_m satisfies the following equality

$$H_m(xy) = \sum_{i=0}^{m} H_i(x)H_{m-i}(y)$$

A higher derivation $\{H_n\}$ of order n is *strong* if H_0 is an identity operator.

Note that a strong higher derivation of order 1 is a derivation.

In [2], E. Posner proved that if the composition d_1d_2 of derivations d_1, d_2 of a prime ring A of characteristic not 2 is a derivation then either $d_1 = 0$ or $d_2 = 0$, and M. Brešer estimated the distance of d_1d_2 to the set of all generalized derivations of A in [1].

In this paper, we show that the Posner's result can be extended to strong higher derivations and estimate the distance of H_i ($i = 1, 2, \cdots, n$) where $\{H_n\}$ is a continuous higher strong derivation of order n.

II. The Results

LEMMA 1. *Let a sequence $\{H_m\}$ be a strong higher derivation of any order on a ring A. Then for all $x, y, z \in A$ and $n \geq 1$,*

$$\begin{aligned} &H_n(xyz) - H_n(xy)z - xH_n(yz) + xH_n(y)z \\ &= \sum_{\substack{i+j=n \\ i,j\geq 1}} H_i(x)yH_j(z) + \sum_{\substack{i+j+k=n \\ i,j,k\geq 1}} H_i(x)H_j(y)H_k(z) \end{aligned}$$

PROOF. For all $x,\ y,\ z \in A$

$$H_n(xyz) = \sum_{\substack{i+j+k=n \\ i,j,k\geq 1}} H_i(x)H_j(y)H_k(z),$$

$$H_n(xy)z = \sum_{\substack{i+j=n \\ i,j,\geq 1}} H_i(x)H_j(y)z$$

and

$$xH_n(yz) = \sum_{\substack{i+j=n \\ i,j\geq 0}} xH_i(y)H_j(z).$$

Thus for all $x,\ y,\ z \in A$,

$$\begin{aligned} &H_n(xyz) - H_n(xyz) - xH_n(yz) + xH_n(y)z \\ &= \sum_{\substack{i+j=n \\ i,j\geq 1}} H_i(x)yH_j(z) + \sum_{\substack{i+j+k=n \\ i,j,k\geq 1}} H_i(x)H_j(y)H_k(z). \end{aligned}$$

The following lemma can be proved by direct computations

LEMMA 2. *Let $f,\ g$ be functions on a ring A. Then for all $x,\ y,\ z,\ w,\ u \in A$,*

$$\begin{aligned} 2f(x)yg(z)wf(u) = &\{f(x)yg(z) + g(x)yf(z)\}wf(u) \\ &+ f(x)y\{g(z)wf(u) + f(z)wg(u)\} \\ &- \{f(x)(yf(z)w)g(u) + g(x)(yf(z)w)f(u)\} \end{aligned}$$

THEOREM 3. *Let $\{H_m\}$ be a continuous strong higher derivation of order n on an ultraprime Banach algebra A with a constant c. Then*

(1) $c||H_1||^2 \leq 4||H_2||$.

(2) $c^2||H_2||\ ||H_1|| \leq 6||H_3|| + \frac{3}{2}||H_1||^3$.

(3) *For each* $1 \le l \le n$,

$$c^2\|H_l\|\,\|H_{n-l}\| \le 6\|H_n\| + \frac{3}{2}\sum_{\substack{i+j+k=n\\ i,j,k\le 1}} \|H_i\|\,\|H_j\|\,\|H_k\| + \frac{3}{2}\sum_{\substack{i+j=n\\ i\ne l,n-l}} \|H_i\|\,\|H_j\|.$$

In particular if $H_2 = 0$ *then* $H_1 = 0$, *and if* $H_3 = 0$ *then*

$$c\|H_1\|^2 \le 4\|H_2\| \le \frac{6}{c^2}\|H_1\|^2.$$

PROOF. By Lemma 1, for every $x, y, z \in A$ and $1 \le l < n$

$$\begin{aligned} &H_n(xyz) - H_n(xy)z - xH_n(yz) + xH_n(y)z \\ &= \sum_{\substack{i+j=n\\ i,j\ge 1}} H_i(x)yH_j(y) + \sum_{\substack{i+j+k=n\\ i,j,k\ge 1}} H_i(x)H_j(y)H_k(z) \\ &= H_l(x)yH_{n-l}(z) + H_{n-l}(x)yH_l(y) + \sum_{\substack{i+j=n\\ i\ne l,n-l}} H_i(x)yH_j(y) \\ &+ \sum_{\substack{i+j+k=n\\ i,j,k\ge 1}} H_i(x)H_j(y)H_k(z), \end{aligned}$$

By Lemma 2, for every $x, y, z, w, u \in A$

$$\begin{aligned} &2\|H_l(x)yH_{n-l}(z)wH_l(u)\| \\ &\le 3(4\|H_n\| + \sum_{\substack{i+j=n\\ i\ne l,n-l}} \|H_i\|\,\|H_j\| + \sum_{\substack{i+j+k=n\\ i,j,k\le 1}} \|H_i\|\,\|H_j\|\,\|H_k\|) \\ &\times \|x\|\,\|y\|\,\|z\|\,\|w\|\,\|u\|\,\|H_l\|, \end{aligned}$$

$$\begin{aligned} &\|M_{H_l}(x), H_{n-l}(z)wH_l(u)\| \\ &\le (6\|H_n\| + \frac{3}{2}\sum_{\substack{i+j=n\\ i\ne l,n-l}} \|H_i\|\,\|H_j\| + \frac{3}{2}\sum_{\substack{i+j+k=n\\ i,j,k\ge 1}} \|H_i\|\,\|H_j\|\,\|H_k\|) \\ &\times \|x\|\,\|z\|\,\|w\|\,\|u\|\,\|H_l\|, \end{aligned}$$

$$
\begin{aligned}
&c||M_{H_l(z)}, H_{n-l}(u)|| \\
&\leq (6||H_n|| + \frac{3}{2} \sum_{\substack{i+j=n \\ i\neq l, n-l}} ||H_i||\,||H_j|| + \frac{3}{2} \sum_{\substack{i+j+k=n \\ i,j,k\geq 1}} ||H_i||\,||H_j||\,||H_k||) \\
&\times ||z||\,||u||,
\end{aligned}
$$

and

$$
\begin{aligned}
&c^2||H_l||\,||H_{n-l}|| \\
&\leq 6||H_n|| + \frac{3}{2} \sum_{\substack{i+j=n \\ i\neq l, n-l}} ||H_i||\,||H_j|| + \frac{3}{2} \sum_{\substack{i+j+k=n \\ i,j,k\geq 1}} ||H_i||\,||H_j||\,||H_k||.
\end{aligned}
$$

The proof of the theorem is completed.

THEOREM 4. *Let $\{H_m\}$ and $\{F_m\}$ be strong higher derivations of any order on a prime algebra A of characteristic not 2. If $\{H_m F_m\}$ is a strong higher derivation of any order, then for each $n \geq 1$ either $H_n = 0$ or $F_n = 0$.*

PROOF. Let $\{H_n F_n\}$ be a strong higher derivation. Since H_1, F_1 and $H_1 F_1$ are derivations, by Porsner's theorem, $H_1 = 0$ or $F_1 = 0$. Since for every x, $y \in A$

$$H_2F_2(xy) = xH_2F_2(y) + H_2F_2(x)y + H_1F_1(x)H_1F_1(y),$$

H_2F_2 is a derivation and so $H_2 = 0$ or $F_2 = 0$. By the induction, $H_n = 0$ or $F_n = 0$ for every $n \geq 0$.

In [2], Posner showed that if D is a derivation on a prime algebra A with $[D(x), x] = 0$ for all $x \in A$, then A is commutative or $D = 0$. We obtain the following theorem from it.

THEOREEM 5. *Let $\{H_n\}$ be a strong higher derivation of order n on a prime ring A. If $[H_n(x),\ x] = 0$ for all $x \in A$ and for every $1 \leq n$, then A is commutative or $H_n = 0$ for every $n \geq 1$.*

PROOF. By Posner's theorem, A is commutative or $H_1 = 0$. If A is not commutative, then H_2 is a derivation with $[H_2 x,\ x] = 0$ for all $x \in A$. Thus $H_2 = 0$. By induction, $H_n = 0$ for every $n \geq 1$.

If $H_n \neq 0$ for some n, there is an i such that $H_1 = H_2 = \cdots = H_{i-1} = 0$ and $H_i \neq 0$. Then H_i is a derivation with $[H_i x, x] = 0$ for all $x \in A$. By Posner's theorem, A is commutative.

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