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Stability of Dynamical Polysystems

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ABSTRACT. We introduce the concept of the prolongation operator and examine some properties of this operator. In c-first countable space X, we prove that each compact subset M of X is stable if and only if $DR(M) = M$ for each polydynamical system.

1. Introduction

N. Kalouptsidis, A. Bacciotti and J. Tsinias have extended the properties of stability reffered to [2] for polydynamical systems when the space is locally compact metric space.

In this paper we introduce the concept of stability for compact set and the general definition of the prolongation operator to characterize stability. In a c-first countable space introduced in [4] which is a general concept than that of a metric space, we prove that each compact subset M of X is stable if and only if $DR(M) = M$ for each polydynamical system.

2. Preliminaries

DEFINITION 2.1. A space X is said to be c-first countable if for each compact subset K of X, there exists a family $\mathcal{U}$ consisting of countably many neighborhoods of K such that every neighborhood of K contains some members of $\mathcal{U}$.

REMARK 2.2. In [4], it was known that every metric space is c-first countable, but the converse does not hold. As for the example, it

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was known that there is a c-first countable and locally compact space which is not a metric space.

In this paper, let X be a locally compact Hausdorff c-first countable space unless otherwise stated, $\mathbb{R}^+$ the set of nonnegative real numbers and 2^X the set of all subsets of X.

DEFINITION 2.3. A dynamical system on X is a continuous map $\pi : X \times \mathbb{R} \to X$ with the following properties:

(a) $\pi(x,0) = x$ for all $x \in X$

(b) $\pi(\pi(x,s),t) = \pi(x, s+t)$ for all $x \in X$ and $s, t \in \mathbb{R}$.

DEFINITION 2.4. A dynamical polysystem on X is a family of dynamical systems $\{\pi_i | i \in I\}$, where I is an arbitrary set of indices.

DEFINITION 2.5. Let $\{\pi_i | i \in I\}$ be a dynamical polysystem on X, Let $x \in X$ and $t \geq 0$. The reachable set from x at time t is a subset $R(x,t) = \{y \in X |$ there exists an integer $n, t_1, \cdots, t_n \in \mathbb{R}^+$ and $i_1, \cdots, i_n \in I$ such that $\sum_{i=1}^{n} t_i = t$ and $y = \pi_{i_n}(\pi_{i_{n-1}}(\cdots \pi_{i_2}(\pi_{i_1}(x,t_1),t_2), \cdots, t_{n-1}), t_n)\}$ of X.

Note that $R(x,0) = \{x\}$ for all $x \in X$. For $A \subset X$ and $t \in \mathbb{R}^+$, we let $R(A,t) = \bigcup_{x \in A} R(x,t)$.

For $S \subset \mathbb{R}^+$ and $x \in X$, we let $R(x,S) = \bigcup_{t \in S} R(x,t)$

DEFINITION 2.6. The reachable map of the polysystem $\{\pi_i | i \in I\}$ is the multivalued map $R : X \times \mathbb{R}^+ \to 2^X$ defined by $R(x,t)$ as the reachable set from x at time t.

DEFINITION 2.7. For $x \in X$, the positive orbit from x for polysystems is a subset $R(x, \mathbb{R}^+)$ of X. Set $R(x, \mathbb{R}^+)$ by $R(x)$. For $A \subset X$, we let $R(A) = \bigcup_{x \in A} R(x)$. Given a dynamical monosystem π, a set $M \subset X$ is called positively invariant if each $x \in M, \gamma^+(x) \subset M$. Now it is natural to define the positively invariance for polysystems $\{\pi_i | i \in I\}$.

DEFINITION 2.8. A subset A of X is positively invariant if $R(A) \subset A$.

3. Stability

DEFINITION 3.1. Let $\{\pi_i | i \in I\}$ be a dynamical polysystem and let M be a compact subset of X. M is *stable* if for any neighborhood U of M, there exists a neighborhood V of M such that $R(V) \subset U$.

DEFINITION 3.2. Let Γ be a multivalued map from X to 2^X. The prolongation operator D transforms Γ into $D\Gamma : X \to 2^X$ defined by $D\Gamma(x) = \{y \in X |$ there exist sequences $x_n \to x, y_n \to y$ such that $y_n \in \Gamma(x_n)\}$.

It is clear that $\Gamma(x) \subset D\Gamma(x)$ for each $x \in X$.

DEFINITION 3.3. For $x \in X$, the set $DR(x)$ is called the prolongation of x.

DEFINITION 3.4. The map $\Gamma : X \to 2^X$ is a c-c map if for any compact $K \subset X$ and $x \in K$ with $\Gamma(x) \not\subset K, \Gamma(x) \cap \partial K \neq \phi$.

All multivalued maps Γ considered in this paper satisfy the reflexivity $x \in \Gamma(x)$ for all $x \in X$.

LEMMA 3.5. *If the map $\Gamma : X \to 2^X$ is a c-c map, then the map $D\Gamma : X \to 2^X$ is a c-c map.*

PROOF. Let K be a compact subset of $X, x \in K$ with $D\Gamma(x) \not\subset K$. Since Γ is a c-c map and $x \in \Gamma(x)$, only two cases are possible: either $\Gamma(x) \cap \partial K \neq \phi$ or $\Gamma(x) \subset \text{Int}K$. First, choose $y \in \Gamma(x) \cap \partial K$. We have $y \in \Gamma(x) \subset D\Gamma(x)$. Thus $D\Gamma(x) \cap \partial K \neq \phi$. Second, $x \in \text{Int}K$. Since $D\Gamma(x) \not\subset K$, there is a $y \in D\Gamma(x) - K$. Therefore, there are sequences $x_n \to x, y_n \to y$ such that $y_n \in \Gamma(x_n)$. We may assume that $x_n \in \text{Int}K, y_n \in X - K$. Since $\Gamma(x_n) \not\subset K$ and Γ is a c-c map, $\Gamma(x_n) \cap \partial K \neq \phi$. Thus there is a sequence $z_n \in \Gamma(x_n) \cap \partial K$. Since ∂K is compact, we choose the sequence $z_n \to z \in \partial K$. We have

$z \in D\Gamma(x)$. It follows that $D\Gamma(x) \cap \partial K \neq \phi$. Hence the lemma is proved.

The following lemma indicates alternate description of the set $DR(x)$.

LEMMA 3.6. *$DR(x) = \bigcap_{U \in \mathcal{N}(x)} \overline{R(U)}$, where $\mathcal{N}(x)$ denotes the set of all neighborhoods of x.*

PROOF. We show first that $DR(x) \subset \bigcap_{U \in \mathcal{N}(x)} \overline{R(U)}$. Let $y \in DR(x)$. Then there are sequences $x_n \to x, y_n \to y$ such that $y_n \in R(x_n)$. For all neighborhoods U, V of x and y, respectively, there is an integer m such that $x_m \in U$ and $y_m \in V$. We have $y_m \in R(x_m) \subset R(U)$ and so $V \cap R(U) \neq \phi$. Therefore $y \in \overline{R(U)}$. It follows that $DR(x) \subset \bigcap_{U \in \mathcal{N}(x)} \overline{R(U)}$. Next, let $y \in \bigcap_{U \in \mathcal{N}(x)} \overline{R(U)}$. Choose a basis at x and y, respectively, (U_n) and (V_n) with $U_n \supset U_{n+1}, V_n \supset V_{n+1}$. For any integer $n, V_n \cap R(U_n) \neq \phi$. Thus there are sequences x_n in U_n and y_n in V_n such that $y_n \in R(x_n)$. Clearly, $x_n \to x$ and $y_n \to y$. Hence we have $y \in DR(x)$. The lemma is proved.

PROPOSITION 3.7. *For every dynamical polysystem :*

1. *$R(x) \subset DR(x)$*
2. *The graph of $DR, G(DR) = \bigcup_{x \in X} \{x\} \times DR(x)$, is closed in $X \times X$*
3. *DR is a c-c map*
4. *For any compact set $M \subset X, DR(M) = \bigcap\{\overline{U} | U$ is positively invariant neighborhoods of $M\}$.*

PROOF. (1) is clear. To prove (2), let $(x, y) \in \overline{G(DR)}$. Then there is a sequence $(x_n, y_n) \in G(DR)$ such that $(x_n, y_n) \to (x, y)$. Since $y_n \in DR(x_n)$, there are sequences $\{x_m^n\}$ and $\{y_m^n\}$ in X such that $x_m^n \to x_n, y_m^n \to y_n$ and $y_m^n \in R(x_m^n)$. Let (U_n) and (V_n) be a basis at x and y with $U_n \supset U_{n+1}, V_n \supset V_{n+1}$, respectively. Then there is an integer n_1 such that $x_{n_1} \in U_1$ and $y_{n_1} \in V_1$. We can choose an integer m_1 so that $x_{m_1}^{n_1} \in U_1$ and $y_{m_1}^{n_1} \in V_1$. Similarly, there is an integer n_2 such that $x_{n_2} \in U_2$ and $y_{n_2} \in V_2$. Also, we can choose an integer m_2 so that $x_{m_2}^{n_2} \in U_2$ and $y_{m_2}^{n_2} \in V_2$. Continuing this process, for any neighborhood U of x, there is an integer k such that $U_k \subset U$. if $i \geq k$, then $x_{m_i}^{n_i} \in U_i \subset U_k \subset U$. Similarly, for any neighborhood V of y, there is an integer k such that $V_k \subset V$. If $i \geq k$, then we have $y_{m_i}^{n_i} \in V_i \subset V_k \subset V$. Therefore, $x_{m_i}^{n_i} \to x$ and $y_{m_i}^{n_i} \to y$. We have $y_{m_i}^{n_i} \in R(x_{m_i}^{n_i})$ and so $y \in DR(x)$. This shows that $(x, y) \in G(DR)$. Hence $D(DR)$ is closed in $X \times X$.

To prove (3) it suffices to show that R is a c-c map. Let K be a compact subset of X and $x \in K$ with $R(x) \not\subset K$. First, let $x \in \partial K$, since $x \in R(x), R(x) \cap \partial K \neq \phi$. Next, let $x \in \text{Int } K$. Choose $y \in R(x) - K$. Then there is a $t \in \mathbb{R}^+$ such that $y \in R(x, t)$. Thus we have $R(x, [0, t]) \cap \text{Int } K \neq \phi$ and $R(x, [0, t]) \cap (X - K) \neq \phi$. Since $R(x, [0, t])$ is path connected, $R(x, [0, t])$ is connected. It follows that $R(x, [0, t]) \cap \partial K \neq \phi$. We clearly have $R(x) \cap \partial K \neq \phi$. Hence R is a c-c map. By the lemma 3.5, DR is a c-c map.

Finally, we will prove (4). Let $y \in DR(M)$. Then there is a $x \in M$ such that $y \in DR(x)$. Thus there are sequences $x_n \to x$ and $y_n \to y$ such that $y_n \in R(x_n)$. Let U be any positively invariant neighborhood of M. Since U is a neighborhood of x, we may assume that $x_n \in U$. As a consequence of positive invariance, $y_n \in R(x_n) \subset U$. Thus $y \in \overline{U}$. Since U is any positively invariant neighborhood of $M, y \in \bigcap \overline{U}$. This shows that $DR(M) \subset \bigcap \overline{U}$. Next, we show that $\bigcap \overline{U} \subset DR(M)$. Let $y \in \bigcap \overline{U}$. Suppose that $y \notin DR(M)$. For all $x \in M, y \notin DR(x)$. By the lemma 3.6, there is a neighborhood U_x of x such that $y \notin \overline{R(U_x)}$. The family $\{U_x | x \in M\}$ is an open cover of M. Since M is compact, there is a finite set $\{x_1, x_2, \cdots, x_n\}$ of M such that $M \subset \bigcup_{i=1}^{n} U_{x_i}$. The set $\bigcup R(U_{x_i})$ is a positively invariant neighborhood of M. Set $U = \bigcup R(U_{x_i})$. We have $\overline{U} = \bigcup \overline{R(U_{x_i})}$ and so $y \notin \overline{U}$. This is a contradiction. Hence we have $y \in DR(M)$. The proof of (4) is completed.

THEOREM 3.8. *For each dynamical polysystem and each compact subset M of X, M is stable if and only if $DR(M) = M$.*

PROOF. Let M be stable. Clearly, we have $M \subset R(M) \subset DR(M)$. Suppose $M \neq DR(M)$. Then there is a point $y \in DR(M) - M$. We can choose $x \in M$ so that $y \in DR(x)$. Thus there are sequences $x_n \to x, y_n \to y$ such that $y_n \in R(x_n)$. Since M is a compact subset of X and $y \notin M$, there exist disjoint neighborhoods U and V of M and y, respectively. By the stability of M, there exists a neighborhood W of M such that $R(W) \subset U$. We choose an integer m so that $x_m \in W, y_m \in V$. Thus we have $y_m \in R(x_m) \subset R(W) \subset U$ and so $U \cap V \neq \phi$. This is a contradiction. Hence $DR(M) = M$.

Conversely, let $DR(M) = M$. Suppose M is not stable. Then there exists a neighborhood U of M such that for any neighborhood V of $M, R(V) \not\subset U$. We can choose a relatively compact neighborhood W of M so that $\overline{W} \subset U$. Let (U_n) be a countable basis of M with $W \supset U_1 \supset U_2 \supset \cdots$. For any n, we have $R(U_n) \not\subset \overline{W}$. Thus there is a sequence x_n in u_n such that $R(x_n) \not\subset \overline{W}$. Since R is a c-c map, $R(x_n) \cap \partial\overline{W} \neq \phi$. Choose $y_n \in R(x_n) \cap \partial\overline{W}$. Since $\partial\overline{W}$ is compact, there is a sequence $y_n \to y$ with $y \in \partial\overline{W}$. Since M is compact, there is a sequence $x_n \to x \in M$. We have $y \in DR(x) \subset DR(M) = M$. This is a contradiction. Hence M is stable. The theorem is proved.

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