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Intermediate Subrings of Normalizing Extensions

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ABSTRACT. Relationships between the prime ideals of a ring R and of a normalizing extension S have been studied by several authors. Relationships between the prime ideals of a ring R and of an intermediate normalizing extension T also have studied by several authors where $R \subset T \subset S$.

In this note, some relationships between prime ideals of T and S are studied.

Suppose that R is a subring of S, sharing the same identity element, and that S is finitely generated as an $R-$ module by elements $a_1, a_2, \cdots, a_n$ with $a_i R = R a_i$. Then S is called a normalizing extension of R. The relationship between the prime ideals of these two rings has been studied by Heinicke, Robson, Lorenz, Passman, Lanski and others [1, 2, 3, 4, 5].

This suggests that a similar relationship could exist between the prime ideals of R and those of any ring T with $R \subset T \subset S$, such a ring T being termed an intermediate normalizing extension of R.

If a ring S is a normalizing extension of a ring R, there is a strong relationship between the sets Spec$S$ and Spec$R$ of prime ideals of these rings. Less clear is the relationship between Spec$S$ and Spec$T$, where T is an intermediate normalizing extension of R.

That question arises as follows.

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If I is a prime ideal of S, $I \cap T$ need not be a prime, nor even a semi-prime, ideal of T. If S is a normalizing extension of T (i.e. if $T = R$), then a sharper result is true ; $I \cap T$ is in fact a semi-prime ideal.

The *incomparability question is this* : if I and I' are distinct primes in S for which $I \cap T$ and $I' \cap T$ have a minimal prime in common, does it follow $I \not\subset I'$ and $I' \not\subset I$? [1].

The quection is still open in general. We will show the answer is *yes* in some case.

Other results herein include new information relating the primes of R, T, and S in some case. For a module M_R or bimodule ${}_RM_R$, $L(M)$ denotes the lattice of submodules or subbimodules of M, and $\mathcal{E}(M)$ the collection of sub(bi)-modules which are essential in M. M_R is a prime module if $M \neq \{0\}$ and $\operatorname{ann}(M') = \operatorname{ann}(M)$ for all nonzero M' in $L(M)$. If M_R is prime, then $P = \operatorname{ann}(M)$ is a prime ideal of R, and we say that M is P–prime.

PROPOSITION 1. *Let P be a prime ideal of R and M_R any module. The followings are equivalent :*

(i) $P = rt\ \operatorname{ann}_R(lt\ \operatorname{ann}_M(P))$.
(ii) $P = rt\ \operatorname{ann}_R(M')$ *for some nonzero* M' *in* $L(M_R)$.
(iii) $P = rt\ \operatorname{ann}_R(X)$ *for some nonempty subset* $X \neq \{0\}$ *of* M [1].

NOTATION : The set of primes of R which occur as annihilators of submodules of M_R will be denoted $\mathrm{rt}M - \mathrm{Spec}R$, and those which occur as annihilators of prime submodules of M make up $\mathrm{ass}(M_R)$

PROPOSITION 2. *Suppose a bimodule ${}_\Lambda M_R$ is decomposed as $M = \sum_1^k \oplus M_i$, where each M_i is in $L({}_\Lambda M_R)$ and is P_i–prime as a right R-*

module. For any E in $\mathcal{E}$ $({}_{\Lambda}M_R)$,

$$\begin{aligned} \mathrm{ass}(E_R) &= rtE - SpecR \\ &= \{P_1, P_2, \cdots, P_k\} \ \ [1]. \end{aligned}$$

DEFINITION 3. Suppose Q is an R-ring (meaning there is a ring homomorphism $R \to Q$ sending 1 to 1) Then Q may be regarded as an R-bimodule. We say that (Q, R) satisfies *the idempotent condition* if Q contains elements $f_1, f_2, \cdots, f_n$ such that :

idem 1 : The f_i are a family of orthogonal idempotents which sum to 1.

idem 2 : Each f_i centralizes R, each $P_i = ann_R(f_i)$ is a prime ideal of R, and $P_i \neq P_j$ where $i \neq j$.

idem 3 : For each i, ${}_R(f_iQ)$ and $(Qf_i)_R$ are P_i-prime modules.

If S is a normalizing extension of R and I a prime ideal of S, then $\bar{S} = S/I$ may be regarded as a prime normalizing extension of $\bar{R} = R/I \cap R$. It was shown in [4] that $(Q(\bar{S}), \bar{R})$ satisfies the idempotent condition,where $Q(\bar{S})$ is the Martindale right ring of quotients of the prime ring $\bar{S}$.

Again suppose S is a normalizing extension of R, and let T be an intermediate subring and J a prime ideal of T. Then T/J may be regarded as an over-ring of $R/R \cap J$.

It was shown in [4] that $(Q(T/J), \bar{R})$ satisfis the idempotent condition, where $Q(T/J)$ is again the Martindale right ring of quotients.

Suppose that $T \neq S$. Choose any ideal I of S maximal with respect to the property that $I \cap T \subseteq J$. Such ideals exist, by Zorn's Lemma, and are prime ideals of S.

$Q(S/I)$ has idempotents $\bar{f}_1, \cdots, \bar{f}_n$ with annihilators in R being $P_1, \cdots P_m$, say. $Q(T/J)$ has idempotens $f'_1, \cdots, f'_k$ whose annihilators in R are, say $P'_1, \cdots, P'_k$. The f'_i are derived from the $\bar{f}'_i$ [4]. In particular, $k \leqslant m$ (after relabelling) $P_i = P'_i$ for $1 \leqslant i \leqslant k$.

The Criterion to tell which of the P_i coming form $Q(\bar{S})$ also occur among the P'_j coming from $Q(T/J)$ is that $\overline{f_iTf_i} \cap \bar{T} \not\subset \bar{J}$ [4].

PROPOSITION 4. *Suppose (Q, R) satisfies the idempotent condition. For any $E \in \mathcal{E}({}_RQ_R)$,*

$$\begin{aligned} \mathrm{ass}(E_R) &= rtE - SpecR \\ &= \{P_1, \cdots, P_n\} \\ &= \mathrm{ass}({}_RE) \\ &= ltE - SpecR \end{aligned}$$

PROOF. ${}_RQ_R = \sum_{j=1}^{n} \oplus Qf_i$ and $(Qf_i)_R$ is P_j-prime therefore

$$\mathrm{ass}(E_R) = \mathrm{rt}E - \mathrm{Spec}R = \{P_1, \cdots, P_n\}$$

by Proposition 2. The other assertions follow by symmetry.

PROPOSITION 5. *Let T be an arbitrary intermediate normalizing extension of R, and J be a prime ideal of T. For any nonzero ideal A of T/J,*

$$\mathrm{ass}(A_R) = rtA - SpecR = \mathrm{ass}({}_RA) = ltA - SpecR = \mathrm{ass}((T/J)_R)$$

and this is a finite set.

PROOF. By the essentiallity Theorem [4], A is essential in $Q = Q(T/J)$. By Proposition 4, each of these sets conicide with $\mathrm{ass}(Q_R)$.

The member of $\mathrm{ass}((T/J)_R)$ are primes of R which are said to be connected to J, and $\mathrm{ass}((T/J)_R)$ is usually called $\mathrm{conn}_R(J)$. By Proposition 5

$$\mathrm{conn}_R(J) = \mathrm{rt}(T/J) - \mathrm{Spec}R$$

i.e., that a prime P of R is connected to J if and only if it is the right annihilator in R of some subset or R–submodule of T/J.

The *incomparability question for* T *and* S is this : Is it possible to have primes $I \subsetneqq I'$ of S and a prime J of T which is a minimal prime over $I \cap T$ and $I' \cap T$?

By passing to S/I and its subrings, we may assume S is prime, and ask if there is a nonzero prime ideal I' of S with $I' \cap T$ contained in some minimal prime J of T.

In fact it is enough to have $I' \cap T \subseteq J$ with I' just a nonzero ideal of S, for we could then enlarge it to an ideal I' of S maximal with respect to having $I' \cap T \subseteq J$ and I' would be prime.

In light of these observations, we may assume that S is prime and J a minimal prime of an intermediate subring T. In this situation there are two distinguished subets of Spec$R$, namely $\mathrm{conn}_R(O_S)$ and $\mathrm{conn}_R(J)$: these being the primes of R connected to the prime ideals O (of S) and J (of T) respectively.

THEOREM 6. *Let S be a primitive normalizing extension of R, T an intermediate subring, and J a minimal prime of T, Then $\mathrm{conn}_R(J) = \mathrm{conn}_R(O_S)$ if and only if for a cyclic J–prime module A_T, A_R contains a submodule which is isomorphic to the direct sum of representatives of isomorphic classes of simple R-submodule of a simple faithful S–module $M = S/K$, where K is a maximal right ideal of S.*

PROOF. Assume that $\mathrm{conn}_R(J) = \mathrm{conn}_R(O_S)$. Let K be a maximal right ideal of S with $(S/K)_S$ is faithful. By [4], $(S/K)_R$ is semisimple R–module and $(S/K)_T$ has finite length. The annihilators of the R-module composition factors comprise $\mathrm{conn}_R(O_s)$. Suppose the annihilators of the T-module composition factors are $I_1, I_2, \cdots, I_t$. The product $I_1 \cdots I_t = 0$ since S/K is faithful. Since $I_1 \cdots I_t = 0 \subset J$ and J is prime, $I_i \subseteq J$. Each I_j is a primitive ideal of T and so prime. Since J is minimal, $I_i = J$. Therefore there is a chain $K_S \subseteq U_T \subset V_T \subseteq S$ and $(U/V)_T$ is a simple module with anihilator J. $\text{conn}_R(J)$ is the set of all composition factors of $(U/V)_R$. Since $\text{conn}_R(J) = \text{conn}_R(O_s), \text{ass}(U/V)_R = \text{ass}((S/K)_R)$. Let $(U/V)_R = A_R$. A_R contains a submodule which is isomophic to a simple R-submodule of $(S/K)_R$ for any simple R-submodule of $(S/K)_R$.

Conversely assume that the condition holds. Then $\text{ass}(A_R) = \text{ass}(S_R)$. This implies that $\text{conn}_R(J) = \text{conn}_R(O_s)$.

Now we are all the details of how to use the primitivity machine [5]. Given a ring S, such that is a certain power series-polynomial ring of the from $S \ll X \gg < Y >$; that is, X and Y are suitally large sets of indeterminate, and elements of S^+ are of the from $\sum_{finite}[\sum s_{\alpha\beta}\alpha]\beta$, where $\alpha's$ and $\beta's$ are monomials in the elements of X and Y, respectively and each $s_{\alpha\beta} \in S$. Elements from one of the sets X, Y commute with those from the other, and with those of S, but not among themselves. For a subset A of S, A^+ consists of all power series-polynomials with coefficients from A.

PROPOSITION 7. *Let S be a nomalizing extension of R and T an intermediate subring.*

(i) *S^+ is a normalizing extension of R^+, and T^+ an intermediate subring.*
(ii) *if $A \triangleleft R$, then $A^+ \triangleleft R^+, A^+ \cap R = A$ and $(R/A)^+ \cong R^+/A^+$.*
(iii) *If A, B are ideals of R, then $A^+B^+ \subseteq (AB)^+$.*
(iv) *A^+ is a primitive ideal if A is prime.*
(v) *$(\cap A_i)^+ = \cap A_i^+$ for any family of subset A_i of S.*
(vi) *If $I \triangleleft R^+$ and $I = lt\ \text{ann}_{R^+}(B)$ for some subset B of R^+, then $(I \cap R)^+ \subseteq I$. Moreover, if I is prime in R^+, then $I \cap R$ is prime in R.*
(vii) *All of (ii)-(vi) apply, with appropriate change to T and S.*

(viii) *If $I \triangleleft S$ and $I \cap R = 0$, then $I^+ \cap R^+ = 0$ [5].*

PROPOSITION 8. *Assume S is prime. J is a minimal prime of T if and only if J^+ is a minimal prime of T^+*

PROOF. Assume that J is a minimal prime of T. T has only a finite number of minimal primes : suppose they are $J_1 \cdots J_k$ with $J = J_1$. Also, the prime radical N of T is nilpotent. From Proposition 7, each J_i^+ is prime and $N^+ = \cap_1^k J_i^+$ is nilpotent. So N^+ is the prime radical of T^+. If $J = J_1^+$ were not minimal, $J_i^+ \subseteq J^+$ for $i \neq 1$. $J_i \subseteq J_1$ which is not possible. J^+ is a minimal prime ideal of R^+.

Conversely, assume that J^+ is a minimal prime ideal of T^+. From Proposition 7, $J^+ \cap R = J$ is prime. T has only a finite member of minimal primes : suppose they are $J_1 \cdots J_k, J_1^+ \cap \cdots \cap J_k^+ = N^+$ is nilpotent. $J_i^+ \subseteq J^+$ for some i. Since J^+ is minimal, $J_i^+ = J^+$. $J_i = J_i^+ \cap R = J^+ \cap R = J$. J is a minimal prime ideal of T.

PROPRSITION 9. *Assume S is prime and let J be a minimal prime of T. $P \in \text{conn}_R(J)$ if and only if $P = \bar{P} \cap R$ for some $\bar{P} \in \text{conn}_{R^+}(J^+)$.*

PROOF. Assume that $P \in \text{conn}_R(J)$. $(T/J)_R$ contains a $P-$prime submodule Y/J which may be assumed cyclic, with $\bar{y} = y+J$ as generator. Consider the R^+-submodule $\bar{y}R^+$ of $Q = Q(T^+/J^+)$. Q_{R^+} decomposes as a direct sum of prime right R^+-module because (Q, R^+) satisfies the idempotent condition. $\bar{y}R^+$ contains a prime submodule $\bar{W}$: this too may be assumed cyclic, with generator $\bar{y}\rho(\rho \in R^+)$. Let rt $\text{ann}_{R^+}(\bar{W}) = \bar{P} \in \text{ass}(T^+/J^+) = \text{conn}_{R^+}(J^+)$.

Now $\bar{y}P = 0$, so $P \subset$ rt $\text{ann}_R(\bar{y}\rho R^+) = R \cap \bar{P}$. To show the opposite inclusion, write ρ as a power series polynomial $\rho = \sum(\sum r_{\alpha\beta}\alpha)\beta$ with $r_{\alpha,\beta}$ in R. Since $\bar{y}P \neq 0, \bar{y}r_{\alpha\beta} \neq 0$ for some α, β. But $\bar{y}\rho(\bar{P} \cap R) = 0$, so $\bar{y}r_{\alpha\beta}(\bar{P} \cap R) = 0$: But $(Y/J)_R$ is $P-$prime, so $\bar{P} \cap R \subseteq P$.

Conversely assume that $\bar{P} \in \text{conn}_{R^+}(J^+), (T^+/J^+)_{R^+}$ contains a $\bar{P}$-prime submodule A/J^+ which may be assumed cyclic, with $\bar{x} = x + J^+$ as generator. Let $x = \sum(\sum s_{\alpha\beta}\alpha)\beta$, $B = \sum s_{\alpha\beta}R$, $C = (B+J)/J$. We will show that $P = \bar{P} \cap R = \text{ann}_R C$. $\bar{x}\bar{P} = 0$. $(s\alpha\beta + J)(\bar{P} \cap R) = 0$ for all α, β. $CP = 0$.

Conversely assume that $Cr = 0$ for $r \in R$. $(x + J^+)R^+r = 0$. This implies that $r \in \bar{P} \cap R = P$. Hence $\text{ann}C = P$. $P \in \text{ass}(R/J)$, $P \in \text{conn}(J)$.

We have immediately following corollary.

COROLLARY 10. *S is prime and let J be a minimal prime of T. $P \in \text{conn}_R(J)$ if and only if $P^+ \in \text{conn}_R(J^+)$.*

PROPOSITION 11. *S is a prime normalizing extension of R and T is an intermediate subring ${}_RR_R$ is essential in ${}_RT_R$. Then incomparability holds.*

PROOF. Let $N(T)$ be The prime radical of T. Since R is a semiprime ring and $N(R) = N(T) \cap R$, $N(T) = 0$. Let $J_1 \cdots J_l$ be the minimal prime ideals of T, where J_i is not essential is ${}_RT_R$. Then $J_1 \cap \cdots \cap J_l = 0$. Let J be a minimal prime ideal of T. $J_1 \cdots J_l \subset J$. $J = J_i$. Every minimal prime ideal of T is not essential in ${}_RT_R$. Therefore incomparalility holds [1].

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