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Lie Algebras with an $\underline{1}$-filtrations

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ABSTRACT. Let L be a Lie algebra over an algebraically closed field F of chracteristic $p > 0$ which has an $\underline{1}$-filtration. We prove that $W(1;\underline{1})$ is the only restricted simple Lie algebra having an $\underline{1}$-filtration. And we show that the even dimensional Lie algebra can not have an $\underline{1}$-filtration.

0. Introduction

Let L be a Lie algebra over an algebraically closed field F of characteristic $p > 0$. If there is a proper subalgebra $L_{(0)}$ of L of codimension 1 such that the filtration $L_{(i)}$ defined by $L_{(0)}$ satisfies the following conditions: $dim(L_{(i)}/L_{(i+1)}) = 1$ for all $i \geq -1$ and there is some r such that $L_{(r)} \neq 0$ but $L_{(r+1)} = 0$, then the set of subalgebras $L_{(i)}$ is called an $\underline{1}$-*filtration* of L defined by $L_{(0)}$. And L is called an $\underline{1}$-filtered Lie algebra[7].

Let L be a $n-$dimensional Lie algebra with an $\underline{1}$-filtration defined by its subalgebra $L_{(0)}$ such that $\dim(L/L_{(0)}) = 1$. Let $\{ L_{(i)} \mid i = -1, 0, 1, ..., n-2 \}$ be the $\underline{1}$-filtration of L with $L_{(-1)} = L$. Then we way assume that each $L_{(i)} = 0$ if $i \geq (n-1)$. Then $dim(L_{(i)}/L_{(i+1)}) = 1$ for each $i = -1, 0, 1, ..., n-2$.

1. Filtered Lie Algebras

LEMMA 1.

$$[x_i, x_j] \begin{cases} \in < x_1, x_2, ..., x_{i+j+1-n} >, & if\, i+j \geq n \\ = 0, & otherwise. \end{cases}$$

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PROOF. By proposition 2.8 in [7],

$$[x_i, x_j] \in [L_{(n-i-1)}, L_{(n-j-1)}] \subset L_{(2n-i-j-2)}.$$

And the property of the $\underline{1}$-filtration implies that if $2n-i-j-2 > n-2$, then $L_{(2n-i-j-2)} = 0$. From $2n-(i+j)-2 > n-1, i+j < n$. Therefore, if $i+j < n$, then $L_{(2n-i-j-2)} = 0$. On the other hand, if $2n-i-j-2 \leq n-2$, then $i+j \geq n$, and $L_{(2n-i-j-2)} \neq 0$. Since $L_{(2n-i-j-2)} =< x_1, x_2, ..., x_{n-(2n-i-j-2)-1} > =< x_1, x_2, ..., x_{i+j+1-n} >$.

In the following Lemma, we will construct a new basis from the given basis of L. Let x_k denote the given basis of L. For each $k, \alpha_{ijk} \in F$ will denote the coefficient of x_k in the product of x_i and x_j expanded with respect to the basis x_k.

LEMMA 2. *Let $x_1, x_2, ..., x_n$ be a basis of L with an $\underline{1}$-filtration defined by a subalgebra $L_{(0)} =< x_1, x_2, ..., x_{n-1} >$. Then there exists a new basis $\{x'_k\}$ such that $[x_k, x_n] = \alpha_{kn(k+1)} x'_{k+1}$ and $< x_1, ..., x_k >=< x'_1, ..., x'_k >$.*

PROOF. We have $[x_1, x_{n-1}] \in [L_{(n-2)}, L_{(0)}] \subset L_{(n-2)} =< x_1 >$ and $[x_1, x_n] \in [L_{(n-2)}, L_{(-1)}] \subset L_{(n-3)} =< x_1, x_2 >$. Let $[x_1, x_{n-1}] = \alpha_{1(n-1)1} x_1$ and let $[x_1, x_n] = \alpha_{1n1} x_1 + \alpha_{1n2} x_2$ for some $\alpha_{1(n-1)1}$, α_{1n1} and $\alpha_{1n2} \in F$. If $\alpha_{1n2} = 0$, then $[x_1, x_n] = \alpha_{1n1} x_1$ and it implies $[x_1, L] \in < x_1 >$. So $x_1 \in L_{(n-1)} = 0$. But x_1 is a basis element and it is not 0. Therefore $\alpha_{1n2} \neq 0$. Now let $x'_1 = x_1$ and

$$x'_2 = \frac{\alpha_{1n1}}{\alpha_{1n2}} x_1 + x_2.$$

Then $[x_1, x_n] = \alpha_{1n1} x_1 + \alpha_{1n2}(-\frac{\alpha_{1n1}}{\alpha_{1n2}} x_1 + x'_1) = \alpha_{1n2} x'_2$.

$$< x_1, x_2 >=< x'_1, x'_2 > .$$

Assume $[x_{k-1}, x_n] = \alpha_{k-1nk}x'_k$ and $< x_1, ..., x_k >=< x'_1, ..., x'_k >$.

$$[x_k, x_n] \in < x'_1, x'_2, ..., x_{k+1} >,$$
$$[x_k, x_n] = (\alpha_{kn1}x'_1 + \alpha_{kn2}x'_2 + ... + \alpha_{knk}x'_k) + \alpha_{kn(k+1)}x_{k+1}.$$

If $\alpha_{kn(k+1)} = 0$, then $[x_k, x_n] \in < x_1, ..., x_k >=< x'_1, ..., x'_k >$.

$$[x_k, L] \in < x'_1, ..., x'_k > .$$

It implies $x_k \in L_{(n-k)}$, and it contradicts to that x_k is not in $L_{(n-k)}$. Therefore, $\alpha_{kn(k+1)} \neq 0$. So we can put

$$x'_{k+1} = [(\alpha_{kn1}x'_1 + \alpha_{kn2}x'_2 + ... + \alpha_{knk}x'_k)/\alpha_{kn(k+1)}] + x_{k+1}.$$

Then $[x_k, x_n] = \alpha_{kn(k+1)}x'_{k+1}$ and $< x_1, ..., x_{k+1} >=< x'_1, ..., x'_{k+1} >$. Therefore, $< x_1, ..., x_n >=< x'_1, ..., x'_n >= L$ and it is a new basis for L such that $[x_k, x_n] = \alpha_{kn(k+1)}x'_{k+1}$.

Next we prove that this new basis of L satisfies the following;

PROPOSITION 3. *If* $L =< x_1, x_2, ..., x_n >$ *with an* $\underline{1}$*-filtration, then* $[x_i, x_j] = \alpha_{ij(i+j+1-n)}x'_{i+j+1-n}$ *for* $i \leq j$ *with* $i + j \geq n$. *i.e.,* $\alpha_{ijk} = 0$ *if* $k < (i + j + 1 - n)$.

PROOF. For$[x_i, x_n]$, by the previous lemma, it is true for each i. Consider $[x_i, x_{n-j}]$ and use an induction on j. If $j = 0$, then $[x_i, x_n] = \alpha_{in(i+1)}x'_{i+1}$ by Lemma 2 and we can also assume that it is true for any $t < j$. Since $[x_i, x_{n-j}] = 0$, for i, j such that $0 \leq t < j \leq i$.

$$[x_i, x_t] = \alpha_{it(i+t+1-n)}x'_{i+t+1-n}.$$

Then consider $[x_i, x_{t+1}] \in < x_1, x_2, ..., x_{i+t+2-n} >$.

$$[x_i, x_{t+1}] = \alpha_{i(t+1)1}x_1 + \alpha_{i(t+1)2}x_2 + ... + \alpha_{i(t+1)(i+t+1-n)}x_{i+t+1-n}$$
$$+ \alpha_{i(t+1)(i+t+2-n)}x_{i+t+2-n}.$$

Now calculate the Jacobi's identity for x_i, x_t, and x_n.

$$[x_i,[x_t,x_n]]+[x_t,[x_n,x_i]]+[x_n,[x_i,x_t]]=0,$$

$$\alpha_{tn(t+1)}[\alpha_{i(t+1)1}x_1+\alpha_{i(t+1)2}x_2+\ldots+\alpha_{i(t+1)(i+t+2-n)}x_{i+t+2-n}]$$

$$+\alpha_{in(i+1)}\alpha_{(i+1)t(i+t+2-n)}x_{i+t+2-n}$$

$$-\alpha_{it(i+t+1-n)}\alpha_{(i+t+1-n)n(i+t+2-n)}x_{i+t+2-n}=0.$$

From this equality, $\alpha_{tn(t+1)}\neq 0$ implies the following:

$$\alpha_{i(t+1)1}=\alpha_{i(t+1)2}=\cdots=\alpha_{i(t+1)(i+t+1-n)}=0.$$

Therefore, $[x_i,x_{t+1}]=\alpha_{i(t+1)(i+t+2-n)}x'_{i+t+2-n}$. For all i,j such that $i+j\geq n$, $[x_i,x_j]=\alpha_{ij(i+j+1-n)}x'_{i+j+1-n}$.

Next, we show that a Lie algebra of dimension p which has an $\underline{1}$-filtration is unique up to isomorphism by using properties of its $\underline{1}$-filtration.

PROPOSITION 4. *Let L be a restricted Lie algebra of dimension p with an $\underline{1}$-filtration. Then there is a basis $\{x_1,x_2,\ldots,x_p\}$ for L which satisfies the following;*

$$[x_i,x_j]=\begin{cases}(j-i)x_{i+j}, & \text{if } i+j\leq p-2\\ 0, & \text{otherwise.}\end{cases}$$

PROOF. Let $\{L_{(i)}\mid i=-1,0,\ldots,p-2\}$ be an $\underline{1}$-filtration of L defined by a subalgebra $L_{(0)}$ of codimension 1. Since $dimL_{(i)}=p-i-1$ for $-1\leq i\leq p-2$, we may assume $L_{(p-2)}$ is generated by one element $b_1\in L$ and $L_{(p-1)}$ is generated by $L_{(p-2)}$ and some element

b_2, i.e. $L_{(p-3)} =< b_1, b_2 >$. For $-1 \leq i \leq p-2$, we can assume $L_{(i)} =< b_1, b_2, ..., b_{p-i-1} >$. Since $b_i \in L_{(p-i-1)}$ but not in $L_{(p-1)}$,

$$[b_i, L] \not\subset L_{(p-i-1)} =< b_1, b_2, ..., b_i >$$
$$[b_i, b_j] \in L_{(p-i-1)}L_{(p-j-1)} \subseteq L_{(2p-i-j-1)}$$
$$\subseteq L_{(2p-(i+p-1)-2)} = L_{(p-i-1)}.$$

By Proposition 3, we can assume that there exists a new basis $\{ x_i \mid -1 \leq i \leq p-2 \}$ such that $[b_i, b_j] = \alpha_{ij(i+j+1-p)} x_{i+j+1-p}$ for i and j such that $i+j \geq p$. Now define this new basis which satisfy the given properties as following:

Let $x_{-1} = b_p$ and $x_i = A_i b_{p-i-1}$, where $A_i = -\frac{(i+1)!}{\prod_{j=1}^{i+1} \alpha_{(p-j)p(p-j+1)}} \in F$, for $i = 0, 1, ..., p-2$. Then we need to show

$$[x_i, x_j] = \begin{cases} (j-i)x_{i+j}, & \text{if } i+j \leq p-2 \\ 0, & \text{otherwise.} \end{cases}$$

First, let's prove $[x_1, x_i] = (i-1)x_{i+1}$

$$\frac{2\alpha_{(p-i-1)(p-2)(p-i-2)}}{\alpha_{(p-1)pp}\alpha_{(p-2)p(p-1)}}$$
$$= \frac{(i-1)(i+2)}{\alpha_{(p-i-2)p(p-i-1)}} \cdot (i-1)(i+2)\alpha_{(p-1)pp}\alpha_{(p-2)p(p-1)}$$
$$- 2\alpha_{(p-i-1)(p-2)(p-i-2)}\alpha_{(p-i-2)p(p-i-1)}$$
$$= (i-1)(i+2)\alpha_{(p-1)pp}\alpha_{(p-2)p(p-1)} - 2\{\alpha_{(p-2)p(p-1)}\alpha_{(p-i-1)(p-1)(p-i-1)}$$
$$+ \alpha_{(p-i-1)p(p-i)}\alpha_{(p-i)(p-2)(p-i-1)}\}$$

(by Jacobi's identity of b_{p-i-1}, b_{p-2}, b_p)

$$= (i-1)(i+2)\alpha_{(p-1)pp}\alpha_{(p-2)p(p-1)} - 2i\alpha_{(p-2)p(p-1)}\alpha_{(p-1)pp}$$
$$- 2\alpha_{(p-i-1)p(p-i)}\alpha_{(p-i)(p-i)(p-i-1)}$$

$$
\begin{aligned}
=&[(i-1)(i+2)-2i]\alpha_{(p-1)pp}\alpha_{(p-2)p(p-1)}-2\{\alpha_{(p-2)p(p-1)}\alpha_{(p-i)(p-1)(p-i)}\\
&\quad+\alpha_{(p-i)p(p-i-1)}\alpha_{(p-i-1)(p-2)(p-i)}\}\\
=&[(i-1)(i+2)-2i-2(i-1)]\alpha_{(p-1)pp}\alpha_{(p-2)p(p-1)}\\
&\quad-2\alpha_{(p-i)p(p-i-1)}\alpha_{(p-i-1)(p-2)(p-i)}
\end{aligned}
$$

By repeating these calculations for i in the second part, we can get the following:

$$
\begin{aligned}
=&[(i-1)(i+2)-2\{i+(i-1)+\ldots+4\}]\alpha_{(p-1)pp}\alpha_{(p-2)p(p-1)}\\
&-2\alpha_{(p-4)(p-2)(p-5)}\alpha_{(p-5)p(p-4)}\\
=&[(i-1)(i+2)-2i+(i-1)+\ldots+5+4+5]\alpha_{(p-1)pp}\alpha_{(p-2)p(p-1)}\\
=&[(i-1)(i+2)-2[\frac{i+1}{2}\cdot i-1]]\alpha_{(p-1)pp}\alpha_{(p-2)p(p-1)}=0.
\end{aligned}
$$

$$
\begin{aligned}
(i-1)x_{i+1}&=-\frac{(i-1)(i+2)!}{\alpha_{(p-1)pp}\cdots\alpha_{(p-i-2)p(p-i-1)}}b_{p-i-2}\\
&=\frac{-2(i+1)!\alpha_{(p-i-1)(p-2)(p-i-2)}}{\alpha_{(p-1)pp}\alpha_{(p-2)p(p-1)}\alpha_{(p-1)pp}\cdots\alpha_{(p-i-1)p(p-i)}}b_{p-i-2}=0.
\end{aligned}
$$

And

$$
[x_1,x_i]=\frac{-2(i+1)!\alpha_{(p-i-1)(p-2)(p-i-2)}b_{p-i-2}}{\alpha_{(p-1)pp}\alpha_{(p-2)p(p-1)}\alpha_{(p-1)pp}\cdots\alpha_{(p-i-1)p(p-i)}}.
$$

Therefore, $[x_1,x_i]=(i-1)x_{i+1}$. Now use an induction on j in $[x_i,x_j]$ and assume $i<j$. Since $-1\le i<j\le p-2$ and $i+j\le p-2$, the initial number for j is zero. Thus, for $j=0,[x_{-1},x_0]$ is the only possible case.

$$
[x_{-1},x_0]=[b_p,A_1b_{p-1}]=-A_1\alpha_{(p-1)pp}b_p=x_{-1}.
$$

Now, assume that $[x_i, x_j] = (j-1)x_{i+j}$. In the case of $j+1$,

$$\begin{aligned} [x_i, x_{j+1}] =&[x_i, \frac{1}{j-1}[x_1, x_j]] = \frac{1}{j-1}[x_i, [x_1, x_j]] \\ =&\frac{1}{j-1}(-[x_1, [x_j, x_i]] - [x_j, [x_i, x_1]]) \\ =&\frac{1}{j-1}((i-j)[x_{i+j}, x_1] + (1-i)[x_{i+1}, x_j]) \end{aligned} \tag{*}$$

If $i+1 < j$, by induction,

$$\begin{aligned} (*) =&\frac{1}{j-1}[(i-j)(1-i-j)x_{i+j+1} + (1-i)(j-i-1)x_{i+j+1}] \\ =&\frac{1}{j-1}(j-1)(j+1-i)x_{i+j+1} = (j+1-i)x_{i+j+1}. \end{aligned}$$

If $i+1 = j$, then

$$[x_i, x_j] = [x_{j-1}, x_j] = x_{2j-1}.$$
$$[[x_i, x_j], x_1] = [x_{2j-1}, x_1] = (1-2j+1)x_{2j}.$$

Therefore,

$$(*) = \frac{1}{j-1}(2-2j)x_{2j} + 0 = -2x_{2j}.$$
$$[x_i, x_{j+1}] = -2x_{2j} = (j+1-i)x_{i+j+1}.$$

Thus we can conclude that

$$[x_i, x_j] = \begin{cases} (j-i)x_{i+j}, & \text{if } i+j \le p-2 \\ 0, & \text{otherwise.} \end{cases}$$

In the previous proposition, we consider in the case of $dimL = p$. For the case of $dimL < p$, we had proved that L is a classical algebra which is embedding in $sl(2, F)$ [7].

COROLLARY 5. *Let L be a Lie algebra satisfying the same hypothesis in proposition 4. Then L is isomorphic to $W(1;\underline{1})$.*

PROOF. By proposition 4, L has a basis x_i such that

$$[x_i, x_j] = \begin{cases} (j-i)x_{i+j}, & \text{if } i+j \leq p-2 \\ 0, & \text{otherwise.} \end{cases}$$

Let $\{e_i\}$ be a standard basis of $W(1;\underline{1})$ and define a map $\varphi : L \mapsto W(1;\underline{1})$ by $\varphi(x_i) = e_i$ for all $i = -1, 0, ..., p-2$.

Then

$$\varphi([x_i, x_j]) = \begin{cases} \varphi((j-i)x_{i+j}) = (j-i)e_{i+j}, & \text{if } i+j \leq p-2 \\ \varphi(0) = 0, & \text{otherwise.} \end{cases}$$

$(j-i)e_{i+j} = [e_i, e_j] = [\varphi(x_i), \varphi(x_j)]$. So $\varphi([x_i, x_j]) = [\varphi(x_i), \varphi(x_j)]$.

By the definition of φ, φ is onto and $ker\varphi = 0$. Thus φ is an isomorphism between L and $W(1:\underline{1})$.

THEOREM 6. *Let L be a Lie algebra over F with a $\underline{1}$- filtration. Then L is isomorphic to $W(1;\underline{m})$.*

PROOF. Let $\{ x_i \mid i = 1, 2, .., p^m \}$ be a basis with a $\underline{1}$-filtration $\{ L_{(i)} \mid i = -1, 0, ..., p^m - 2 \}$ of L, where $L_{(i)}$ is generated by $x_1, x_2, ...,$ x_{n-i-1}. Since $W(1;\underline{m})$ has a basis $\{ e_i \mid i = -1, 0, ..., p^m - 2 \}$ satisfying

$$[e_i, e_j] = \begin{cases} (j-i)e_{i+j}, & \text{if } i+j \leq p^m - 2 \\ 0, & \text{otherwise.} \end{cases}$$

We defined a correspondence between $\{ e_i \mid i = -1, ..., p^m - 2 \}$ and $\{ x_i \mid i = 1, ..., p^m \}$ by

$$e_i \mapsto \frac{(i+1)!}{\prod_{j=1}^{i+1} \alpha_{(p-j)p(p-j+1)}} x_{p^m - i - 1},$$

for each i. By the similar proof in the previous corollary, φ is an isomorphism from $W(1;\underline{m})$ to L.

THEOREM 7. *If L is an even dimensional Lie algebra, then L can not have any $\underline{1}$-filtration.*

PROOF. Suppose there is an $\underline{1}$-filtration $\Im(L)$ of L.

$$\Im(L) : 0 = L_{(n-1)} \subset L_{(n-2)} \subset ... \subset L_{(1)} \subset L_{(0)} \subset L_{(-1)} = L.$$

Then $dim(L_{(i)}/L_{(i+1)}) = 1$ for each i. Let $\{x_1, x_2, ..., x_n\}$ be a basis of L such that $L_{(i)} =< x_1, x_2, ..., x_{n-i-1} >$ for $i = -1, 0, ..., n-2$ and $L_{(n-1)} = 0$. Let $n = 2k$ for some integer k. Since $[x_{k-1}, [x_k, x_n]] + [x_k, [x_n, x_{k-1}]] + [x_n, [x_{k-1}, x_k]] = 0$,

$$\alpha_{kn(k+1)}\alpha_{(k-1)(k+1)1}x_1 + \alpha_{(k-1)nk} \cdot 0 + 0 = 0$$
$$\alpha_{kn(k+1)}\alpha_{(k-1)(k+1)1} = 0.$$

Since $x_k \in L_{n-k-1}$ and $x_n \in L_{-1}$, $[x_k, x_n] \in L_{n-k-1}L_{-1} \subseteq L_{n-k-2}$ which is generated by $x_1, x_2, ..., x_{k+1}$. If $[x_k, x_n] \in< x_1, ..., x_k >$, i.e., $\alpha_{kn(k+1)} = 0$.

$$[x_k, L] \in< x_1, ..., x_k >\subseteq L_{n-k-1}.$$

So $x_k \in L_{n-k}$, but $x_k \notin L_{n-k} =< x_1, ..., x_{k-1} >$. Therefore, $\alpha_{kn(k+1)} \neq 0$. It implies that $\alpha_{kn(k+1)}\alpha_{(k-1)(k+1)1} = 0$

$$(**) \qquad \alpha_{(k-1)(k+1)1} = 0.$$

From the Jacobi's identity for x_{k-j-1}, x_{k+j}, x_n, we will prove $\alpha_{(k-j)(k+j)1} = 0$. If $j = 1$, then

$$[x_{k-2}, [x_{k+1}, x_n]] + [x_{k+1}, [x_n, x_{k-2}]] + [x_n, [x_{k-2}, x_{k+1}]] = 0,$$
$$\alpha_{(k+1)n(k+2)}\alpha_{(k-2)(k+2)1}x_1 + \alpha_{(k-2)n(k-1)}\alpha_{(k-1)(k+1)1}x_1 = 0$$

By (**), $\alpha_{(k+1)n(k+2)}\alpha_{(k-2)(k+2)1} = 0$. Using the similar argument, $\alpha_{(k+1)n(k+2)} \neq 0$, thus $\alpha_{(k-2)(k+2)1} = 0$.

Now we assume $\alpha_{(k-j+1)(k+j-1)1} = 0$.

$$[x_{k-j}, [x_{k+j-1}, x_n]] + [x_{k+j-1}, [x_n, x_{k-j}]] + [x_n, [x_{k-j}, x_{k+j-1}]] = 0,$$
$$\alpha_{(k+j-1)n(k+j)}\alpha_{(k-j)(k+j)1}x_1 + \alpha_{(k-j)n(k-j+1)}\alpha_{(k-j+1)(k+j-1)1}x_1 = 0.$$

By hypothesis, $\alpha_{(k-j+1)(k+j-1)1} = 0$,

$$\alpha_{(k+j-1)n(k+j)}\alpha_{(k-j)(k+j)1} = 0.$$

Since $\alpha_{(k+j-1)n(k+j)} \neq 0$, $\alpha_{(k-j)(k+j)1} = 0$. Therefore, $\alpha_{(k-j)(k+j)1} = 0$ for all $j = 1, 2, ..., k-1$.i.e.,

$$\alpha_{1(n-1)1} = \alpha_{2(n-2)1} = ... = \alpha_{(k-1)(k+1)1} = 0$$

From the Jacobi's identity for x_i, x_{n-1}, x_n $(i = 1, 2, .., n-2)$, if $i = 1$, then

$$\alpha_{(n-1)nn} + \alpha_{2(n-1)2} - \alpha_{1(n-1)1} = 0.$$

Assume $\alpha_{(n-1)nn} + \alpha_{i(n-1)i} - \alpha_{(i-1)(n-1)(i-1)} = 0$ for $i-1$. To prove it for i, consider the following;

$$[x_i, [x_{n-1}, x_n]] + [x_{n-1}, [x_n, x_i]] + [x_n, [x_i, x_{n-1}]] = 0,$$
$$\alpha_{in(i+1)}\{\alpha_{(n-1)nn} + \alpha_{(i+1)(n-1)(i+1)} - \alpha_{i(n-1)i}\} = 0.$$

Since $\alpha_{in(i+1)} \neq 0$, for each $i = 1, 2, ..., n-2$,

(***) $$\alpha_{(n-1)nn} + \alpha_{(i+1)(n-1)(i+1)} - \alpha_{i(n-1)i} = 0.$$

If $i = n-2$, then $\alpha_{(i+1)(n-1)(i+1)} = \alpha_{(n-1)(n-1)(n-1)} = 0$. Thus $\alpha_{(n-1)nn} = \alpha_{(n-2)(n-1)(n-2)}$.

$$\alpha_{(n-1)nn} + \alpha_{(n-2)(n-1)(n-2)} - \alpha_{(n-3)(n-1)(n-3)} = 0$$

implies

$$2\alpha_{(n-1)nn} = \alpha_{(n-3)(n-1)(n-3)}.$$

Using this fact in $(***)$ for all i,

$$(n-2)\alpha_{(n-1)nn} = \alpha_{1(n-1)1}.$$

From the previous induction, $\alpha_{(n-1)nn} = 0$ and $[x_{n-1}, L] \subseteq L_{(0)}$. So $x_{n-1} \in L_{(1)}$. It contradicts to the definition of $L_{(1)}$. Therefore, L has no $\underline{1}$-filtration when $dimL$ is even.

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