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# Flows Associated with Semiflows

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## 1. Introduction

It is known that if the map $\pi^t : X \to X$ is surjective for each $t \in \mathbb{R}^+$, then a triple $(X_\infty, \mathbb{R}, \pi_\infty)$ is flow [2]. The purpose of this paper is to prove this fact without assumption, as follows:

THEOREM. *Let* $(X, \mathbb{R}^+, \pi)$ *be a semiflows. Then a triple* $(X_\infty, \mathbb{R}, \pi_\infty)$ *induced by* $(X, \mathbb{R}^+, \pi)$ *satisfy the conditions of flow.*

We say that a triple $(X, T, \pi)$ is a *dynamical system*, where $X$ is a topological space, $T$ is a topological group and $\pi : X \times Y \to X$ is a continuous function satisfying (a) $\pi(x, e) = x$ for $x \in X$ and (b) $(\pi(\pi(x,t), s) = \pi(x, ts)$ for any $s, t \in T$ and $x \in X$. For convenience we shall write $x\pi t$ or $xt$ for $\pi(x,t)$. We call a *flow* if $T = \mathbb{R}$, the group of reals, and a *semiflow* if $T = \mathbb{R}^+$, the group of nonnegative reals, and a *discrete* if $T = \mathbb{Z}$, the group of integers.

## 2. Proof of the theorem

In this section we shall show our theorem. To prove the theorem, we need some definitions and lemmas. Let $(X, \mathbb{R}^+, \pi)$ be a semiflow and let $\tilde{X} = \prod_{t \in \mathbb{R}^+} X_t$ be the product space over the index set $s \in \mathbb{R}^+$ directed by the usual order relation on the reals and $X_t = X$ for all $t \in \mathbb{R}^+$. Given $x = (x_t) \in \tilde{X}$ and $s \in \mathbb{R}^+$, define the map $\tilde{\pi} : \tilde{X} \times \mathbb{R}^+ \to \tilde{X}$ by $\tilde{\pi}((x_t), s) = (x_t \pi s)$. Then it is known that a triple $(\tilde{X}, \mathbb{R}^+, \tilde{\pi})$ is a semiflow which is usually called the direct product of

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semiflow family $\{(X_t,\pi) : X_t = X, t \in \mathbb{R}^+\}$. Put $X_\infty = \{x = (x_t) \in \tilde{X} : x_t = x_s\pi(s-t)\ s,t \in \mathbb{R}^+ \text{ with } s > t\}$. Let $\pi_\infty : X_\infty \times \mathbb{R} \to X_\infty$ be defined by: given $x = (x_t) \in X_\infty$ and $r \in \mathbb{R}^+$,

$$x\pi_\infty r = \begin{cases} (y_t), & y_t = x_0\pi(r-t), 0 \le t \le r, \\ (y_t), & y_t = x_{t-r}, t \ge r \end{cases}$$

and

$$x\pi_\infty(-r) = (y_t), \quad y_t = x_{t-r}.$$

Let us show that this definition is well-defined.

LEMMA 1. *The map $\pi_\infty$ is well-defined.*

PROOF. It is sufficient to show that $(x\pi_\infty r)\pi_s(s-t) = (x\pi_\infty r)_t$ for $s,t \in \mathbb{R}^+$ with $s > t$. To prove this claim, we have four cases.

$(a)$ $0 \le r \le t$. From $(x\pi_\infty r)_s\pi(s-t) = x_{s-r}\pi(s-t) = x_{t-r}$ and $(x\pi_\infty r)_t = x_{t-r}$ , we obtain $(x\pi_\infty r)_s\pi(s-t) = (x\pi_\infty r)_t$.

$(b)$ $t \le r \le s$. From $(x\pi_\infty r)_s\pi(s-t) = x_{s-r}\pi(s-t) = (x_{s-r}\pi(s-r))\pi(r-t) = x_0\pi(r-t)$ and $(x\pi_\infty r)_t = x_0\pi(r-t)$, we obtain $(x\pi_\infty r)_s\pi(s-t) = (x\pi_\infty r)_t$.

$(c)$ $s \le r$. From $(x\pi_\infty r)_s\pi(s-t) = (x_0\pi(r-s)\pi(s-t) = x_0\pi(r-t)$ and $(x\pi_\infty r)_t = x_0\pi(r-t)$, we obtain $(x\pi_\infty r)_s\pi(s-t) = (x\pi_\infty r)_t$.

$(d)$ $r \le 0$. From $(x\pi_\infty r)_s\pi(s-t) = x_{s-r}\pi(s-t) = x_{t-r}$ and $(x\pi_\infty r)_t = x_{t-r}$ ,we obtain $(x\pi_\infty r)_s\pi(s-t) = (x\pi_\infty r)_t$. Hence we conclude that $(x\pi_\infty)_s\pi(s-t) = (x\pi_\infty r)_t$.

DEFINITION 2. $P_l : X_\infty \to X_l, \quad l \in \mathbb{R}^+$, is called *canonical projection* .

In order to show that a triple $(X_\infty, \mathbb{R}, \pi_\infty)$ is flow, we need the following property:

LEMMA 3. *The family* $\{P_t^{-1}(U) \cap X_\infty : t \in \mathbb{R}^+, U \subset X, open\}$ *is a basis for the topology on* $X_\infty$.

PROOF. Since $X_\infty$ is a subspace of product space $\tilde{X}$, given an open neighborhood $W_\infty$ of a point $x = (x_t) \in X_\infty$, there exist open neighbord $U_1, U_2, \ldots, U_n$ of $x_{t_1}, x_{t_2}, \ldots, x_{t_n}$, respectively, such that $x = (x_t) \in P_{t_1}^{-1}(U_1) \cap P_{t_2}^{-1}(U_2) \cap \cdots \cap P_{t_n}^{-1}(U_n) \cap X_\infty \subset W_\infty$ for $t_1, t_2, \ldots, t_n \in \mathbb{R}^+$. To prove this claim, it is sufficient to prove that $x = (x_t) \in P_s^{-1}(V) \cap X_\infty \subset W_\infty$ for $s \in \mathbb{R}^+$ and an open neighborhood $V$ of $x_s$ in $X$. Put $s = \max_{1 \leq i \leq n}\{t_i\}$. Then, we have $s \geq t_i$. By the continuity of semiflows, we find an open neighborhood $V$ of $x_s$ with $V\pi(s - t_i) \subset U_i$ for an open neighborhood $U_i$. From the fact that $y_{t_i} = y_s\pi(s - t_i) \in V\pi(s - t_i) \subset U_i$ for $y \in P_s^{-1}(V) \cap X_\infty$, we get $y \in P^{-1}t_i(u_i) \cap X_\infty$. Consequently, we have

$$\begin{aligned} x = (x_t) \in & P_s^{-1}(V) \cap X_\infty \\ & \subset P_{t_1}^{-1}(U_1) \cap P_{t_2}^{-1}(U_2) \cap \cdots \cap P_{t_n}^{-n}(U_n) \cap X_\infty \subset W_\infty, \end{aligned}$$

ending the proof.

If we replaces index set $\mathbb{R}^+$ by index set $\mathbb{Z}^+$, the statements of Lemma 3 remain true.

COROLLARY 4. *The family* $\{P_n^{-1}(U) \cap X_\infty : n \in \mathbb{Z}^+, U \subset X, open\}$ *is a basis for the topology on* $X_\infty$.

PROOF. For an open set $W_\infty$ in $X_\infty$, let $x \in W_\infty \subset X_\infty$. By Lemma 3, there exist a $t_1 \in R^+$ and an open subset $U_1$ in $X$ such that $x \in P_{t_1}^{-1}(U_1) \cap X_\infty \subset W_\infty$. In particular, we choose $n \in \mathbb{Z}^+$ with $n > t_1$. The fact that $V\pi(n - t_1) \subset U_1$ for an open neighborhood $V$ of $x_n$ comes from the fact $x_s\pi(n - t_1) = x_{t_1} \in U_1$ and the continuity of semiflow. If $y_n \in V$, it follows that $y_{t_1} = y_n\pi(n - t_1) \in V\pi(n - t_1) \in$

$U_1$. So, we obtain

$$x \in P_n^{-1}(V) \cap X_\infty \subset P_{t_1}^{-1}(U_1) \cap X_\infty \subset W_\infty,$$

as desired.

PROOF. To prove our Theorem, it is sufficient to prove the fact that the map $\pi_\infty : X_\infty \times \mathbb{R} \to X_\infty$ satisfy the conditions of flow.

First condition. From $(x\pi_\infty 0)_t = x_t$, we obtain $x\pi_\infty 0 = x$.

Second condition. we prove that $(x\pi_\infty r)\pi_\infty s = x\pi_\infty(r+s)$ for any $r, s \in \mathbb{R}$.

Case (1) $r \geq 0, s \geq 0$.

(a) $0 \leq t \leq s$. The fact that $((x\pi_\infty r)\pi_\infty s)_t = (x\pi_\infty(r+s))_t$ comes from the fact $((x\pi_\infty r)\pi_\infty s)_t = (x\pi_\infty r)_0\pi(s-t) = (x_0\pi r)\pi(s-t) = x_0\pi(s-t)$ and $(x\pi_\infty(r+s))_t = x_0\pi(r+s-t)$. Hence, we obtain $(x\pi_\infty r)\pi_\infty s = x\pi_\infty(r+s)$.

(b) $s \leq t \leq r+s$. The fact that $((x\pi_\infty r)\pi_\infty s)_t = (x\pi_\infty(r+s))_t$ comes from the fact $((x\pi_\infty r)\pi_\infty s)_t = (x\pi_\infty r)_{t-s} = x_0\pi(r-t+s)$ and $(x\pi_\infty(r+s))_t = x_0\pi(r+s-t)$. Hence, we obtain $(x\pi_\infty r)\pi_\infty s = x\pi_\infty(r+s)$.

(c) $r+s \leq t$. The fact that $((x\pi_\infty r)\pi_\infty s)_t = (x\pi_\infty(r+s))_t$ comes from the fact that $((x\pi_\infty r)\pi_\infty s)_t = (x\pi_\infty r)_{t-s} = x_{t-s}$ and $(\pi_\infty(r+s))_t = x_{t-s-r}$. Hence, we obtain $(x\pi_\infty r)\pi_\infty s = x\pi_\infty(r+s)$.

The proofs of the other cases (2) $r \geq 0, -r \geq s \geq 0$, (3) $r \geq 0, s \leq -r$, (4) $r \leq 0, s \leq 0$, (5 ) $r \leq 0, s \geq -r$, and (6) $r \leq 0, 0 \leq s \leq -r$,respectively, are similar to the proof of (1).

Third condition. We shall show that $\pi_\infty : X_\infty \times \mathbb{R} \to X_\infty$ is continuous. Let us write $x\pi_\infty r = y \in X_\infty$ for the corresponding map $\pi_\infty : X_\infty \times \mathbb{R} \to X_\infty$. By Lemma3, the basis neighborhoods of $y$ always have the form $P_s^{-1}(U) \cap X_\infty$ for $s \in \mathbb{R}^+$. We claim that $y_s = x_n\pi(n+r-s)$ for $n = s+|x|+1$. To prove this claim, we have to show the following three cases.

(i) $0 \le s \le r$. $y_s = x_0\pi(r-s) = (x_n\pi_n)\pi(r-s) = x_n\pi(n-s+r)$.

(ii) $0 \le r \le s$. From the fact that $y_s = x_{s-r}$ and $n = s+r+1 > s-r$, we obtain $y_s = x_{s-r} = x_n\pi(n-s+r)$.

(iii) $r < 0$. From the fact that $y_s = x_{s-r}$ and $n = s-r+1 > s-r$, we obtain $y_s = x_{s-r} = X_n\pi(n-s+r)$.

By (i), (ii), (iii), we have always the form $y_s = x_n\pi(n-s+r)$. Now, there exists an open neighborhood $V$ of $x_n$ and $0 < \varepsilon < 1$ , respectively, such that $V\pi(n+r-s-\varepsilon, n+r-s+\varepsilon) \subset U$ by the continuity of semiflows. From $x_n \in V$, we assert that $(P_n^{-1}(V) \cap X_\infty)\pi_\infty(r-\varepsilon, r+\varepsilon) \subset P_s^{-1}(U) \cap X_\infty$ for $x \in P_n^{-1}(V) \cap X_\infty$. To prove this fact, we shall that $(z\pi_\infty t)_s = z_n\pi(n+t-s)$ for any $z \in P_n^{-1}(V) \cap X_\infty$ and $t \in (r-\varepsilon, r+\varepsilon)$. To check this claim, we have three cases.

($\alpha$) $0 \ge s \ge t.(z\pi_\infty t)_s = z_0\pi(t-s) = (z_n\pi_n)\pi(t-s) = z_n\pi(n+t-s)$.

($\beta$) $0 \ge t \ge s$. By $(z\pi_\infty t)_s = z_{s-t}$ and $n \le s-t$, we have $(z\pi_\infty t)_s = z_n\pi(n+t-s)$.

($\gamma$) $0 \le s$. By $(z\pi_\infty t)_s = z_{s-t}$ and $n > s-t$, we have $(z\pi_\infty t)_s = z_n\pi(n+t-s)$

Therefore, we obtain that $(z\pi_\infty t)_s = z_n\pi(n+t-s) \in V\pi(n+r-s-\varepsilon, n+r-s+\varepsilon)$. Consequently, from the fact that $z\pi_\infty t \subset P_s^{-1}(U) \cap X_\infty$,the map $\pi_\infty : X_\infty \times \mathbb{R} \to X_\infty$ is continuous. This completes the proof.

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