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A Characterization of The Strong Measurability via Oscillation

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ABSTRACT. Let (Ω, Σ, μ) be a measure space. A function $f : \Omega \to X$ is said to be *equioscillated* if for each set $A \in \Sigma$ of positive measure and for each $\epsilon > 0$, there is a measurable subset B of A of positive measure such that the inequality

$$\sup_{\omega \in B} x^* f(\omega) - \inf_{\omega \in B} x^* f(\omega) < \epsilon$$

holds for every x^* with $\|x^*\| \leq 1$. Strong measurability of a vector valued function is characterized using equioscillation.

Let (Ω, Σ, μ) be a finite measure space, and let X be a Banach space with continuous dual X^*.

A function $f : \Omega \to X$ is said to be *strongly measurable* if there exists a sequence (f_n) of simple functions from Ω into X such that

$$\lim_n \|f_n - f\| = 0$$

μ-almost everywhere.

A strongly measurable function $f : \Omega \to X$ is called *Bochner integrable* if there exists a sequence (f_n) of simple functions from Ω into X such that

$$\lim_n \int_\Omega \|f - f_n\| d\mu = 0.$$

A function $f : \Omega \to X$ is *Pettis integrable* if

(1) for every $x^* \in X^*$, the scalar function $x^* f$ is μ-measurable and μ-integrable, and

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(2) for every $A \in \Sigma$, there exists an $x_A \in X$ such that

$$x^*(x_A) = \int_A x^* f d\mu$$

for every $x^* \in X^*$.

In [3], L. H. Riddle and E. Saab introduced the Bourgain property to investigate the Pettis integrability of a function $f : \Omega \to X^*$.

In this paper, we strengthen the Bourgain property to, what we call, the equioscillatedness. Surprisingly, we find that the strong measurability of a function $f : \Omega \to X$ is equivalent to saying that f is equioscillated. This is a characterization of strong measurability of vector valued functions analogous to Pettis's measurability theorem.

All notations and notions used and not defined in this paper can be found in [1], [2], and [3].

DEFINITION 1. Let (Ω, Σ, μ) be a measure space. A family Ψ of real-valued functions on Ω is said to have the *Bourgain property* if the following condition is satisfied : For each set $A \in \Sigma$ of positive measure and for each $\epsilon > 0$, there is a finite collection $\mathcal{F}$ of measurable subsets of A of positive measure such that for each $f \in \Psi$, the inequality

$$\sup_{\omega \in B} f(\omega) - \inf_{\omega \in B} f(\omega) < \epsilon$$

holds for some member B of $\mathcal{F}$. A function $f : \Omega \to X$ is said to have the Bourgain property if the set $\{x^* f : \|x^*\| \leq 1\}$ has the Bourgain property.

PROPOSITION 2 [3, Theorem 13]. *A bounded function $f : \Omega \to X^*$ which has the Bourgain property is Pettis integrable.*

The converse is not always true. In fact, there is a universally Pettis integrable function without the Bourgain property [3, Example 14].

DEFINITION 3. Let (Ω, Σ, μ) be a measure space. A family Ψ of real-valued functions on Ω is said to be *equioscillated* if the following condition is satisfied : For each set $A \in \Sigma$ of positive measure and for each $\epsilon > 0$, there is a measurable subset B of A of positive measure such that the inequality

$$\sup_{\omega \in B} f(\omega) - \inf_{\omega \in B} f(\omega) < \epsilon$$

holds for every function f in Ψ. A function $f : \Omega \to X$ is said to be *equioscillated* if the set $\{x^* f : \|x^*\| \leq 1\}$ is equioscillated.

The difference between the equioscillatedness and the Bourgain property lies in the number of elements of $\mathcal{F}$. For the equioscillatedness, we required $\mathcal{F}$ to be a singleton $\{B\}$.

PROPOSITION 4. *If a family Ψ of real-valued functions on Ω is equioscillated, then the pointwise closure of Ψ is also equioscillated.*

PROOF. Let $\epsilon > 0$ be given, and let A in Σ be fixed with $\mu(A) > 0$. Since Ψ is equioscillated, there exists a subset B of A with $\mu(B) > 0$ such that

$$\sup_{\omega \in B} f(\omega) - \inf_{\omega \in B} f(\omega) < \frac{\epsilon}{2} \tag{1}$$

for all f in Ψ. For each g in $\overline{\Psi}$, there exists a net (f_α) in Ψ such that g is a pointwise limit of (f_α). We can choose f_{α_0} in $\{f_\alpha\}$ and ω_1, ω_2 in B such that

$$|\sup_{\omega \in B} g(\omega) - f_{\alpha_0}(\omega_1)| < \frac{\epsilon}{4} \quad \text{and} \quad |\inf_{\omega \in B} g(\omega) - f_{\alpha_0}(\omega_2)| < \frac{\epsilon}{4}. \tag{2}$$

By (1), (2)

$$\sup_{\omega \in B} g(\omega) - \inf_{\omega \in B} g(\omega) < \epsilon.$$

This completes the proof.

The following is stated in p.42 of [2, Corollary 3] without an indication of an argument. To make this paper self-contained and clear, we give an argument.

LEMMA 5. *A function $f : \Omega \to X$ is strongly measurable if and only if f is the μ-almost everywhere uniform limit of a sequence of countably valued strongly measurable functions.*

PROOF. We need to prove only sufficiency. Let $g_n = \sum_{k=1}^{\infty} x_{n,k}\mathcal{X}_{E_{n,k}}$ with $x_{n,k} \in X$, $E_{n,i} \cap E_{n,j} = \phi$ for $i \neq j, E_{n,k} \in \Sigma$ and $\cup_k E_{n,k} = \Omega$. Suppose that f is the μ-almost everywhere uniform limit of (g_n). We may assume that

$$\|g_n(\omega) - f(\omega)\| < \frac{1}{n} \tag{3}$$

for all $w \in \Omega$ without loss of generality. Since $\mu(\Omega)$ is finite, for each n, there exists an integer l_n such that $\mu(\cup_{k=l_n+1}^{\infty} E_{n,k}) < \frac{1}{n}$. Put $E_n = \cup_{k=l_n+1}^{\infty} E_{n,k}$. This gives rise to a simple function

$$f_n = \sum_{k=1}^{l_n} x_{n,k}\mathcal{X}_{E_{n,k}}.$$

Note that

$$f_n = g_n \quad \text{on} \quad \Omega - E_n. \tag{4}$$

Now we take a subsequence of (f_n) as follows: Pick a subsequence $\{E_{n_k}\}$ of $\{E_n\}$ so that

$$\mu(E_{n_k}) < \frac{1}{2^k},$$

and let

$$\widetilde{E_k} = \cup_{p=k}^{\infty} E_{n_p}.$$

Then clearly, $\{\widetilde{E_k}\}$ is decreasing and

$$\mu(\widetilde{E_k}) \leq \frac{1}{2^k} + \frac{1}{2^{k+1}} + \cdots = \frac{1}{2^{k-1}}.$$

Let $A = \cap_{k=1}^{\infty} \widetilde{E_k}$. Then $\mu(A) = 0$. For every $\omega \notin A$, there exists an integer n such that $\omega \notin \widetilde{E_n}$, which implies $\omega \notin \widetilde{E_k}$ for all $k > n$ since the sequence $\{\widetilde{E_k}\}$ is decreasing. Moreover, for such ω and n,

$$\begin{aligned} \|f_{n_k}(\omega) - f(\omega)\| &\leq \|f_{n_k}(\omega) - g_{n_k}(\omega)\| + \|g_{n_k}(\omega) - f(\omega)\| \\ &= \|g_{n_k}(\omega) - f(\omega)\| \\ &< \frac{1}{n_k} \end{aligned}$$

using (3) and (4). This shows that the sequence of simple functions (f_{n_k}) converges to f on $\Omega - A$. This completes the proof of the lemma.

THEOREM 6. *A function* $f : \Omega \to X$ *is strongly measurable if and only if* f *is equioscillated.*

PROOF. Suppose $f : \Omega \to X$ is strongly measurable and let (f_n) be a sequence of simple functions for which

$$\lim_n \|f - f_n\| = 0 \quad a.e.$$

Let A be a measurable subset of Ω with $\mu(A) > 0$, and let $\epsilon > 0$ be given. By Egoroff's theorem, there exists a set B with $\mu(\Omega - B) < \mu(A)$ such that the sequence (f_n) converges uniformly to f on B. Choose an integer n so that

$$\|f(\omega) - f_n(\omega)\| < \frac{\epsilon}{4}$$

for all ω in B. Let $f_n = \sum_{i=1}^{m} x_i \mathcal{X}_{E_i}$, $x_i \in X$, and let $\cup_{j=1}^{m} E_j = \Omega$. Then there exists a set C in $\{E_1, E_2, \cdots, E_m\}$ such that

$$\mu(A \cap B \cap C) > 0$$

since $\mu(A \cap B) > 0$, and $\mu(A \cap B) = \mu(A \cap B \cap (\cup_{j=1}^{m} E_j)) = \sum_{j=1}^{m} \mu(A \cap B \cap E_j)$.

Let x^* be an element of $\{x^* \in X^* : \|x^*\| < 1\}$. Then for all ω_1, ω_2 in $A \cap B \cap C$,

$$\begin{aligned} |x^*f(\omega_1) - x^*f(\omega_2)| &\leq |x^*f(\omega_1) - x^*f_n(\omega_1)| + |x^*f_n(\omega_1) - x^*f_n(\omega_2)| \\ &\quad + |x^*f_n(\omega_2) - x^*f(\omega_2)| \\ &\leq \|f(\omega_1) - f_n(\omega_1)\| + 0 + \|f_n(\omega_2) - f(\omega_2)\| \\ &< \frac{\epsilon}{2}. \end{aligned}$$

Now we can choose ω_1, ω_2 in $A \cap B \cap C$ so that

$$|\sup_{\omega \in A \cap B \cap C} x^*f(\omega) - x^*f(\omega_1)| < \frac{\epsilon}{4}$$

and

$$|\inf_{\omega \in A \cap B \cap C} x^*f(\omega) - x^*f(\omega_2)| < \frac{\epsilon}{4}.$$

Then

$$\sup_{\omega \in A \cap B \cap C} x^*f(\omega) - \inf_{\omega \in A \cap B \cap C} x^*f(\omega) \leq x^*f(\omega_1) + \frac{\epsilon}{4} - (x^*f(\omega_2) - \frac{\epsilon}{4}) < \epsilon$$

This shows that f is equioscillated.

Conversely, suppose f is equioscillated. Since (Ω, Σ, μ) is a finite measure space, for each n, there exists a sequence $(B_{n,k})$ of pairwise disjoint measurable sets with positive measure such that

$$\sup_{\omega \in B_{n,k}} x^*f(\omega) - \inf_{\omega \in B_{n,k}} x^*f(\omega) < \frac{1}{n}, \quad n = 1, 2, 3, \cdots$$

for each x^* in $\{x^* \in X^* : \|x^*\| \leq 1\}$, and $\mu(\Omega - \cup_k B_{n,k}) = 0$.

Note that $\mu(\Omega - \cap_n \cup_k B_{n,k}) = 0$. Define

$$f_n = \sum_{k=1}^{\infty} f(\omega_{n,k}) \mathcal{X}_{B_{n,k}}$$

where each $\omega_{n,k}$ is fixed in $B_{n,k}$.

If ω is an element of $\cap_n \cup_k B_{n,k}$ then there exists a sequence (B_{n,k_n}) such that ω is in B_{n,k_n} for each n. And

$$\begin{aligned}
\|f_n(\omega) - f(\omega)\| &= \sup_{\|x^*\| \leq 1} | x^* f_n(\omega) - x^* f(\omega) | \\
&= \sup_{\|x^*\| \leq 1} | x^* f(\omega_{n,k_n}) - x^* f(\omega) | \\
&\leq \sup_{\|x^*\| \leq 1} | \sup_{\omega \in B_{n,k_n}} x^* f(\omega) - \inf_{\omega \in B_{n,k_n}} x^* f(\omega) | \\
&< \frac{1}{n}
\end{aligned}$$

for all ω in $\cap_n \cup_k B_{n,k}$.

This shows that (f_n) converges uniformly to f almost everywhere, hence f is strongly measurable by Lemma 5. This completes the proof.

COROLLARY 7. *A bounded function* $f : \Omega \to X$ *is Bochner integrable if and only if* f *is equioscillated.*

COROLLARY 8. *A function* $f : \Omega \to X^*$ *is strongly measurable if and only if* $\{xf : \|x\| \leq 1\}$ *is equioscillated.*

PROOF. For each x^{**} in X^{**} with $\|x^{**}\| \leq 1$, there exists a net (x_α) in $\{x \in X : \|x\| \leq 1\}$ which converges weak* to x^{**} by Goldstine's theorem. Hence $x^{**}f$ belongs to the pointwise closure of $\{xf : \|x\| \leq 1\}$ since $((f(\omega)) (x_\alpha))$ converges to $x^{**}(f(\omega))$ for each ω in Ω.

By Proposition 4, $\{x^{**}f : \|x^{**}\| \leq 1\}$ is equioscillated if and only if $\{xf : \|x\| \leq 1\}$ is equioscillated. Hence f is strongly measurable by Theorem 6.

There exists a function $f : \Omega \to X$ such that $x^*f = 0$ almost everywhere for each x^* in X^*, but f is not equioscillated (i.e. f is not strongly measurable).

EXAMPLE 9. Let $(\Omega, \Sigma, \mu) \equiv ([0,1],$ Lebesgue measurable sets, Lebesgue measure) and let $l_2[0,1]$ be the set of all functions $x : [0,1] \to R$ for which $\|x\| = [\sum_{r \in [0.1]} |x(r)|^2]^{\frac{1}{2}} < \infty$. Then $l_2[0,1]$ is a Banach space whose dual is $(l_2[0,1])^* = l_2[0,1]$. The action of Ψ in $(l_2[0,1])^*$ on x in $l_2[0,1]$ is given by

$$\Psi(x) = \sum_{r \in [0,1]} \Psi(r)x(r).$$

Define a function $f : \Omega \to l_2[0,1]$ by $r \to e_r$, where

$$e_r(t) = \begin{cases} 1 & \text{if} \quad r = t \\ 0 & \text{otherwise.} \end{cases}$$

For each x^* in $(l_2[0,1])^*$, $x^*f : \Omega \to R$ has at most countably many non-vanishing points, since $x^*f(r) = x^*(r)$ for all r in Ω, and since x^* is an element of $l_2[0,1] = (\ l_2[0,1]\ )^*$. Hence $x^*f = 0$ almost everywhere.

But f is not equioscillated. Indeed, for B in Σ with $\mu(B) > 0$, fix r in B and put $x^* = e_r$ then

$$x^*f(r) = e_r(f(r)) = e_r(e_r) = \sum_{t \in [0,1]} e_r(t)e_r(t) = 1.$$

And if $t(\neq r)$ is an element of B, then

$$x^*f(t) = e_r(f(t)) = e_r(e_t) = \sum_{s \in [0.1]} e_r(s)e_t(s) = 0.$$

Hence

$$\sup_{\omega\in B} x^* f(\omega) - \inf_{\omega\in B} x^* f(\omega) = 1.$$

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