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On SF-Rings and Semisimple Rings

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ABSTRACT. In this note, we study conditions under which SF-rings are semi-simple. We prove that left SF-rings are semisimple for each of the following classes of rings: (1) left non-singular rings of finite rank; (2) rings whose maximal left ideals are finitely generated; (3) rings of pure global dimension zero and (4) rings which is pure-split. Also it is shown that left SF-rings without zero-divisors are semisimple.

Let R be an associative ring with identity. A ring R is called a (left)*SF-ring* if every simple left R-module is flat. It is known that R is regular if and only if every left R-module is flat. In connection with this fact, Ramamurthi [7] bigan the study of the relation of SF-rings and regular rings. M.B. Rege[8], Yue Chi Ming [11, 12] and J. Chen [3] proved that the SF property implies the regularity for each of the following classes of rings: (1) semi-local rings; (2) rings finitely generated as modules over their centers; (3) quasi-duo rings; (4) left p.p. rings; (5) left semi-artinian rings; (6) left non-singular rings of finite Goldie dimension.

In this note, we prove that left SF-rings are semisimple for each of the following classes of rings: (1) left non-singular rings of finite rank; (2) rings whose maximal left ideals are finitely generated; (3) rings of pure global dimension zero; (4) rings which is pure-split. Also, it is shown that left SF-rings without zero-divisors are semisimple.

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Throughout this paper, R represents an associative ring with identity and every R-module is unital. We say that R is *semisimple* whenever ${}_RR$ is a semisimple left R-module; equivalently, all left R-modules are projective. Also, it is known that R is a semisimple ring if and only if all simple left R-modules are projective. For left R-modules A and C, an epimorphism $f : A \to C$ is called *pure(finitely split)* if $\mathrm{Hom}_R(M, A) \to \mathrm{Hom}_R(M, C)$ is an epimorphism for every finitely presented (finitely generated) left R-module M. A left R-module M is *finitely projective* if every epimorphism onto M is finitely split and *pure-projective* if every pure epimorphism onto M is split. A left *annihilator ideal* in a ring R is any ideal which equals left annihilator ideal of some subset of R. As usual, $l(S)$ denotes the left annihilator ideal of S in R.

We first need the following proposition and lemma.

PROPOSITION 1 [9]. *Let R be a subring of a ring S and M a left R-module. If M is flat and the left S-module $S \otimes_R M$ is finitely projective over S, then M is finitely projective.*

LEMMA 2. *Let R be a subring of a ring S. If every flat S-module is finitely projective, then the same holds for every flat R-module.*

PROOF. Let M be a flat R-module. Then for any monomorphism of S-modules $A \to B$, the natural homomorphism $A \otimes_R M \to B \otimes_R M$ is a monomorphism. Since $A \otimes_S (S \otimes_R M) \simeq A \otimes_R M$ and $B \otimes_S (S \otimes_R M) \simeq B \otimes_R M$, $S \otimes_R M$ is flat over S. Hence $S \otimes_R M$ is finitely projective by the hypothesis. So M is a finitely projective R-module by Proposition 1.

Recall that a ring R has a *finite left rank* (equivalently, finite Goldie dimension) if there are no infinite direct sums of nonzero left ideals within R. Every left noetherian ring has a finite left rank.

THEOREM 3. *A left nonsingular SF-ring of finite left rank is semisimple.*

PROOF. If R is a left nonsingular SF-ring of finite rank, then R is a subring of the maximal left quotient ring Q of R. By Theorem 12.2.5 [10], Q is semisimple. Since every flat Q-module is finitely projective, every flat left R-module is also finitely projective by Lemma 2. Therefore, every simple left R-module is projective, and so R is semisimple.

A left R-module is called *R-Mittag Leffler (R-ML)* if the canonical homomorphism $\mu_{M,I} : R^I \otimes M \to M^I$ defined by $\mu_{M,I}(\{r_i\} \otimes m) = \{r_i m\}$ is a monomorphism for every set I. So M is finitely presented if and only if M is finitely generated and R-ML. A ring R is of *left pure global dimension zero* if every left R-module is pure-projective [2]. Also in [2], M is called *pure-split* if every pure submodule of M is a direct summand of M. In the following theorem, we can see that every simple flat R-module is projective if R is pure-split.

THEOREM 4. *The following conditions are equivalent:*

(1) *R is semisimple.*

(2) *R is a left SF-ring and every simple left R-module is R-ML.*

(3) *R is a left SF-ring and every simple left R-module is finitely presented.*

(4) *R is a left SF-ring whose maximal left ideals are finitely generated.*

(5) *R is a left SF-ring with pure global dimension zero.*

(6) *R is a pure-split left SF-ring.*

PROOF. Since a module is finitely presented if and only if it is finitely generated and R-ML, the implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) follows. For every maximal left ideal M, R/M is a simple left R-module, so it is finitely presented. Hence M is finitely generated. Thus (3) $\Rightarrow$ (4) is shown. To prove (4) $\Rightarrow$ (1), let S be a simple left R-module. Then $S \simeq R/l(S)$ is flat and finitely presented since $l(S)$ is a maximal left ideal. By Corollary 11.5[10], S is projective. Thus R is semisimple. The implication (1) $\Rightarrow$ (5) is obvious.

(5) $\Rightarrow$ (2). Every simple left R-module is pure-projective, so it is finitely pure-projective. Since finitely pure-projective modules coincide with R-ML modules (see [6]), it follows that every simple left R-module is R-ML.

(1) $\Rightarrow$ (6). Over a semisimple ring R, every R-module is pure-split since every exact sequence is split. Thus R is a pure-split SF-ring.

(6) $\Rightarrow$ (1). Since every simple left R-module S is flat, every maximal left ideal is a pure submodule of R. Hence it is a direct summand of R. Since $l(S)$ is a maximal left ideal of R, the sequence $0 \to l(S) \to R \to S \to 0$ is split exact. Thus every simple left R-module S is projective and hence R is semisimple.

PROPOSITION 5. *Let R be a SF-ring without zero-divisors. Then R is semi-simple.*

PROOF. Let S be a simple left R-module and x a nonzero element of S. Then S is flat and so it is torsion-free in the sense that $x \neq 0$ and s not a zero-divisor implies $sx \neq 0$. Hence $Rx = S$ is isomorphic to R and so it is projective.

REMARKS. (1) As we have seen, SF-rings whose flat modules are finitely (or, singly) projective is semisimple. Thus left Noetherian SF-rings and Prüfer SF-rings are also semisimple by Proposition 15 and 18 of [1]. Semiperfect SF-rings are also semisimple since every finitely generated flat module is projective over semiperfect rings.

(2) A ring R is called a left semi-artinian ring if every nonzero left R-module has nonzero socle. Chen ([3]) shows that a semi-artinian

SF-ring is (von Neumann) regular. Therefore, from [5] we can see that the following conditions are equivalent for a left semi-artinian ring R: (i) R is a SF-ring; (ii) R is regular; (iii) R is an f-V-ring; (iv) R is fully left idempotent.

(3) A ring without non-zero nilpotent elements is called a *reduced ring*. By Rege [8], it is proved that a reduced SF-ring is strongly regular. From this fact, it follows that commutative SF-rings are regular. Moreover, commutative SF-rings are V-rings (rings over which all simple modules are injective), because a simple module is flat if and only if it is injective over a commutative ring.

(4) Azumaya[1] conjectured that every flat left R-module is finitely projective if (and only if) $l(a_1) \subset l(a_1a_2) \subset \cdots$ terminates for every sequence $a_1, a_2, \cdots$, in R. In connection with this, we suggest a question whether SF-rings satisfying the above condition on termination of ascending chains are semisimple. We also point out that if R satisfies the above condition on termination of ascending chains then R has no infinite number of orthogonal idempotents and that reduced SF-rings with no infinite number of orthogonal idempotents are semisimple (by the above remark (3) and Corollary 2.16 [4]).

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