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On a New Selection Theorem(*)

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ABSTRACT. The purpose of this note is to give a new selection theorem which is an essential tool for proving the new kind of existence theorem of the equilibrium price comparable to the Debreu-Gale-Nikaido theorem.

The Debreu-Gale-Nikaido theorem [1] is a potential tool to prove the existence of a market equilibrium price. Walras' law is of a quantitative nature (i.e. it measures the value of the total excess demand), and it is interesting to note that the existence result holds true under some qualitative assumptions. In fact, the Debreu-Gale-Nikaido theorem states that the continuity of the excess demand function and Walras' law has the following implication : For some price and corresponding value of the excess demand function, it is not possible to respond with a new price system such that the value at the new price of every element in the value of the demand function associated with the old price system is strictly positive.

In recent years, several infinite-dimensional generalizations of the classical Debreu-Gale-Nikaido theorem have been proved by Florenzano [4], Mehta-Tarafdar [6], Toussaint [7], Yannelis [8] and Kim-Rim [5]. In the previous generalizations [4-8] of the Debreu-Gale- Nikaido theorem, the price set lies in either a Banach space or a locally convex linear topological space E and it is assumed that the positive cone P

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has an interior point e. Also the excess demand value $\xi(p)$ at the price p is non-empty compact convex. In fact, in [4,6] they used Alaoglu's theorem to obtain the weak* compact convexness of the price simplex.

Here we will investigate to relax the compact convex assumption on the correspondence ξ by assuming the additional condition on the price set (i.e. the price set X might be compact convex in the real price market). Under those settings, we shall need a new selection theorem for proving a new kind of existence theorem of the equilibrium price.

In this note, we will give a new selection theorem which is an essential tool for proving the new kind of existence theorem of the equilibrium price comparable to the Debreu-Gale-Nikaido theorem.

We first recall the following notation and definition. Let A be a non-empty set. We shall denote by 2^A the family of all non-empty subsets of A. Let X, Y be two Hausdorff linear topological spaces and $\xi : X \to 2^Y$ be a correspondence. Then ξ is said to be *upper hemicontinuous* at $x \in X$ whenever x is in the upper inverse of an open set so is a neighborhood of x.

We now prove the following new continuous selection theorem:

LEMMA. *Let X be a non-empty compact convex set in a locally convex Hausdorff linear topological space E, E^* a dual space of E, and $<,>$ denote the dual pairing on $E^* \times E$. Let $\xi : X \to 2^{E^*}$ be an upper hemicontinuous correspondence such that for each $x \in X$,*

$$\Psi(x) := \{z \in X \mid Re < y, z > > 0 \quad \text{for all} \quad y \in \xi(x)\} \quad \text{is non-empty.}$$

Then there exists a continuous selection $f : X \to X$ of Ψ, i.e. $f(x) \in \Psi(x)$ for each $x \in X$.

PROOF. Let $x_o \in X$ be arbitrarily given. Then, by the assumption, there exists a point $z_o \in \Psi(x_o)$ and z_o determines an open half space containing $\xi(x_o)$. Since ξ is upper hemicontinuous, there exists an open neighborhood $N(x_o)$ of x_o such that for each $x \in N(x_o)$, $\xi(x)$ is contained in the open half space determined by z_o; and hence $z_o \in \Psi(x)$ for all $x \in N(x_o)$.

Since $\{N(x_o) \mid x_o \in X\}$ is an open covering of the compact set X, there exists a finite subcovering $\{N(x_i) \mid i \in I$ where I is a finite set $\}$ of the covering $\{N(x_o) \mid x_o \in X\}$. Then we can find a corresponding set of finite points $\{z_i \mid i \in I\}$ such that $z_i \in \Psi(x)$ for all $x \in N(x_i)$. Also there exists a continuous partition of unity $\{\alpha_i \mid i \in I\}$ subordinated to this subcovering $\{N(x_i) \mid i \in I\}$.

Now we define a function $f : X \to X$ by

$$f(x) := \sum_{i \in I} \alpha_i(x) \cdot z_i \quad \text{for each } x \in X.$$

Since $\Psi(x)$ is convex for each $x \in X$, f is the desired continuous selection of Ψ. In fact, for any $x \in X$ with $x \in N(x_{i_1}) \cap \cdots \cap N(x_{i_k})$ for some $\{i_1, \cdots, i_k\} \subset I$, $\{z_{i_1}, \cdots, z_{i_k}\} \subset \Psi(x)$ and $f(x) = \sum_{i \in I} \alpha_i(x) \cdot z_i = \sum_{j=1}^{k} \alpha_{i_j}(x) \cdot z_{i_j} \in \Psi(x)$. This completes the proof.

There have been many continuous selection theorems on closed convex valued lower hemicontinuous correspondences or generalized Fan-Browder type correspondences, but our lemma is a new selection result on upper hemicontinuous correspondence.

Using the Tychonoff fixed point theorem, we can obtain the following existence theorem in compact convex settings which is comparable to the Debreu-Gale-Nikaido theorem.

THEOREM. *Let X be a non-empty compact convex subset in a locally convex Hausdorff linear topological space E, E^* a dual space of E, and $<,>$ denote the dual pairing on $E^* \times E$. Let $\xi : X \to 2^{E^*}$ be an upper hemicontinuous correspondence such that for each $x \in X$, there exists a point $z \in \xi(x)$ such that $Re < z, x > \leq 0$.*

Then there exists a price vector $\hat{x} \in X$ satisfying the following: For each $y \in X$, there exists $z_y \in \xi(\hat{x})$ such that $Re < z_y, y > \leq 0$.

PROOF. Suppose the conclusion were false. Then for every $x \in X$,

$$\Psi(x) := \{y \in X \mid Re < z, y > > 0 \text{ for all } z \in \xi(x)\}$$

is non-empty. By the Lemma, there exists a continuous selection $f : X \to X$ of Ψ, i.e. $f(x) \in \Psi(x)$ for each $x \in X$. So we have $Re < z, f(x) > > 0$ for all $z \in \xi(x)$.

By applying the Tychonoff fixed point theorem to f, there exists a point $\hat{x} \in X$ such that $f(\hat{x}) = \hat{x}$. Therefore we have $Re < z, \hat{x} > > 0$ for all $z \in \xi(\hat{x})$, which contradicts the assumption. This completes the proof.

REMARKS. (1) As we noted before, the compactness assumption on the price set X is a meaningful assumption in real market economy. And the assumption on the excess demand correspondence ξ is weaker than the previous corresponding results in [4,6]. In fact, each non-empty value $\xi(p)$ need not be compact nor convex.

(2) In the above conclusion, if the excess demend $z \in \xi(\hat{x})$ is commonly determined for every $y \in X$, then the conclusion can be written as follows:

There exists a price vector $\hat{x} \in X$ such that $\xi(\hat{x}) \cap X^o \neq \emptyset$, where

$$X^o = \{z \in E^* \mid Re < z, x > \leq 0 \text{ for all } x \in X\} \text{ is the polar of } X.$$

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