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Noncommutative Versions of Singer-Wermer Theorem

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ABSTRACT. In this paper,we show that if A is a Banach algebra with radical R and D is a left derivation on A then $D(A) \subset R$ if and only if $Q_R D^n$ is continuous for all $n \geq 1$,where Q_R is the canonical quotient map from A onto A/R.

1. Introduction

In 1955 Singer and Wermer proved that the range of a continuous derivation on a commutative Banach algebra is contained in the Jacobson radical[4]. In the same paper they conjectured that the assumption of continuity is not necessary. In 1988 Thomas proved the Singer-Wermer conjecture[5]. There are also non-commutative versions of the Singer-Wermer theorem. For example, in [1] it was proved that every continuous left derivation on a Banach algebra A maps A into its radical. It is the purpose of the present paper to generalize it.

2. Preliminaries

Throughout, A will represent an algebra over a complex field C and R the Jacobson radical. Let X and Y be Banach spaces, and let T be a linear map from X into Y. Then the separating space of T is defined as

$$S(T) = \{y \in Y : \text{there exists } \ x_k \to 0 \ \text{ in } \ X \ \text{ with } \ T(x_k) \to y\}.$$

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A linear mapping $D : A \to A$ is called a derivation if

$$D(xy) = xD(y) + D(x)y \quad (x, y, \in A).$$

A linear mapping $D : A \to A$ is called a left derivation if

$$D(xy) = xD(y) + yD(x) \quad (x, y \in A).$$

Also Q_I will denote the canonical quotient map from A onto A/I where I is any closed (2-sided)ideal of A, and N the set of all positive integers.

3. Main Results

LEMMA 3.1. *Let D be a left derivation on an algebra A. Then*

$$D^n(xy) = \sum_{r=0}^{n-1} \binom{n-1}{r} [D^r(x)D^{n-r}(y) + D^r(y)D^{n-r}(x)] \qquad (n \in N, x, y \in A).$$

PROOF. We prove the statement by induction. It is trivial when $n = 1$. We assume that

$$D^n(xy) = \sum_{r=0}^{n-1} \binom{n-1}{r} [D^r(x)D^{n-r}(y) + D^r(y)D^{n-r}(x)].$$

Then

$$\begin{aligned} D^{n+1}(xy) =& D(D^n(xy)) \\ =& \sum_{r=0}^{n-1} \binom{n-1}{r} [D^r(x)D^{n+1-r}(y) + D^{n-r}(y)D^{r+1}(x) \\ &+ D^r(y)D^{n+1-r}(x) + D^{n-r}(x)D^{r+1}(y)] \end{aligned}$$

$$
\begin{aligned}
=&xD^{n+1}(y)+\sum_{r=1}^{n-1}\binom{n-1}{r}D^r(x)D^{n+1-r}(y)\\
&+\sum_{r=1}^{n-1}\binom{n-1}{n-r}D^r(x)D^{n+1-r}(y)+D^n(x)D(y)\\
&+yD^{n+1}(x)+\sum_{r=1}^{n-1}\binom{n-1}{r}D^r(y)D^{n+1-r}(x)\\
&+\sum_{r=1}^{n-1}\binom{n-1}{n-r}D^r(y)D^{n+1-r}(x)+D^n(y)D(x)\\
=&xD^{n+1}(y)+\sum_{r=1}^{n-1}[\binom{n-1}{r}+\binom{n-1}{n-r}]D^r(x)D^{n+1-r}(y)\\
&+D^n(x)D(y)+yD^{n+1}(x)\\
&+\sum_{r=1}^{n-1}[\binom{n-1}{r}+\binom{n-1}{n-r}]D^r(y)D^{n+1-r}(x)+D^n(y)D(x)\\
=&xD^{n+1}(y)+\sum_{r=1}^{n-1}\binom{n}{r}D^r(x)D^{n+1-r}(y)\\
&+D^n(x)D(y)+yD^{n+1}(x)\\
&+\sum_{r=1}^{n-1}\binom{n}{r}D^r(y)D^{n+1-r}(x)+D^n(y)D(x)\\
=&\sum_{r=0}^{n}\binom{n}{r}[D^r(x)D^{n+1-r}(y)+D^r(y)D^{n+1-r}(x)].
\end{aligned}
$$

LEMMA 3.2. *Let D be a left derivation on a Banach algebra A and P a primitive ideal of A. If there exists a constant $C>0$ such that $\|Q_pD^n\|\leq C^n$ for all $n\in N$,then $D(P)\subset P$.*

PROOF. Note that the quotient algebra A/P is a primitive algebra and is hence semisimple. Let $x\in A$ and $y\in P$ and observe that

$$xD(y)=D(xy)-yD(x)\in D(P)+P.$$

This shows that $D(P) + P$ is a left ideal of A, hence $Q_P(D(P))$ is a left ideal of A/P. A simple modification of the proof of Lemma 2.1 in [3] shows that $Q_P(D^n(x^n)) = Q_P(n!(D(x))^n)$ holds for all $x \in P$ and all $n \in N$. Hence, by the assumption, we have

$$\begin{aligned} \|(Q_P(D(x)))^n\|^{1/n} &= (n!)^{-1/n}\|Q_P(D^n(x^n))\|^{1/n} \\ &\le (n!)^{-1/n}C\|x^n\|^{1/n} \to 0 \end{aligned}$$

as $n \to \infty$. This shows that $Q_P(D(P))$is a quasinilpotent left ideal of A/P, therefore, it is contained in the radical of A/P. Semisimplicity forces $D(P) \subset P$.

Now we prove our main theorem.

THEOREM 3.3. *Let D be a left derivation on a Banach algebra A. Then $D(A) \subset R$ if and only if $Q_R D^n$ is continuous for all $n \in N$.*

PROOF. If $D(A) \subset R$, then for all $n \in N, Q_R D^n = 0$ which is continuous. Conversely, define the linear subspace

$$K = \{x \in R : D^n(x) \in R \quad \text{for all} \quad n \in N\}.$$

Then K is closed as a consequence of the continuity of each $Q_R D^n (n \in N)$. Let $x \in A$ and $y \in K$ and note that

$$\begin{aligned} D^n(xy) =& xD^n(y) + yD^n(x) \\ &+ \sum_{r=1}^{n-1} \binom{n-1}{r} [D^r(x)D^{n-r}(y) + D^r(y)D^{n-r}(x)] \end{aligned}$$

by Lemma 3.1. Then it is easy to see that $D^n(xy) \in R$ for all $n \in N$. This shows that K is a left ideal of A. A similar argument shows that it is a right ideal. Consequently, K is a closed(2-sided) ideal of A.

Let $y \in S(D)$. Then there exists $x_k \to 0$ with $D(x_k) \to y$. But, for every $n \in N, Q_R D^n(y) = \lim_{k \to \infty} Q_R D^{n+1}(x_k) = 0$ since each $Q_R D^{n+1}$ is continuous. Therefore we know that $D^n(y) \in R$ for all $n \in N$. So $S(D) \subset K$. Since $D(K) \subset K$,we may define a continuous linear map $\widetilde{D} : A/K \to A/K$ by $\widetilde{D}(x + K) = D(x) + K(x \in A)$. This shows that we may also define a map

$$\Phi \widetilde{D}^n Q_K : A \to A/K \to A/K \to A/R$$

by $\Phi \widetilde{D}^n Q_K(x) = Q_R D^n(x)(x \in A)$ where Φ is the canonical inclusion map from A/K onto A/R (which exists since $K \subset R$). We therefore conclude that $\|Q_R D^n\| \leq \|\widetilde{D}\|^n$ for all $n \in N$, since the other maps are norm depressing. By Lemma 3.2, $D(R) \subset R$ since R is the intersection of the primitive ideals of A. Then we may define a left derivation $\bar{D} : A/R \to A/R$ by $\bar{D}(x + R) = D(x) + R(x \in A)$. By the result of B.E.Johnson [2] zero is the only derivation of a commutative semisimple Banach algebra. Thus,if A/R is commutative, then $\bar{D} = 0$ since A/R is semisimple. On the other hand, if A/R is noncommutative, then $\bar{D} = 0$ by Proposition 1.6 [1] since A/R is prime. In any case $\bar{D} = 0$. Consequently, we see that $D(A) \subset R$.

The following corollary is due to Brešar [1].

COROLLARY 3.4. *Let D be a continuous left derivation on a Banach algebra A. Then $D(A) \subset R$.*

PROOF. Since D is continuous, $Q_R D^n$ is continuous for all $n \in N$. Theorem 3.3 shows that $D(A) \subset R$.

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